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FINITE ELEMENTS, FINITE ROTATIONS AND SMALL STRAINS OF FLEXIBLE SHELLS

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Abstract—The concept of finite elements for the analysis of shells is developed here with several important advances.

Firstly, the Kirchhoff theory of shells is refined to include a transverse shear deformation. The refined theory admits simpler approximating functions while preserving continuity at the inter-sections of elements.

Secondly, the motion of the element is decomposed into a rigid body motion followed by a deformation. The decomposition serves to extend existing formulations for linearly elastic elements to problems involving finite rotations and buckling.

Thirdly, the Lagrange equations are introduced to derive the equations of the discrete system. The method yields the consistent inertial terms for any manner of motion, oscillatory or transient.

Finally, the simplest approximating polynomials are introduced in the context of the shear-deformation theory. Further simplification is achieved by the introduction of constraints analogous to the Kirchhoff hypothesis of the continuum theory. The constraints provide a rational basis for neglecting the contribution of transverse shear in the strain energy. The resulting approximation converges rapidly to the Kirchhoff theory for examples cited.

NOTATION

The usual suffix notations are used to indicate the components of tensors. Latin minuscules represent the numbers 1, 2, and 3 while Greek minuscules represent 1 and 2 except in the Appendix where the special meaning is stated. The summation convention applies to minuscules. Latin majuscules signify a particular particle of the medium.

The arrow ($\rightarrow$) over a symbol denotes a vector and a caret ($\hat{}$) denotes a unit vector. The vertical line (|) signifies covariant differentiation with respect to the undeformed-surface coordinates. A comma (,) denotes partial differentiation.

The initial and convected coordinate lines and tangent base vectors are shown in Fig. 3.

The symbol $\doteq$ is used to indicate an approximation of a previous equation.

Some basic notations follow:

θ^α	surface coordinate
θ^3	length along the normal to the undeformed reference surface
$\vec{a}_\alpha(\vec{A}_\alpha)$	base vectors tangent to undeformed (deformed) surface–coordinate line θ^α, c.f. equations (1a), (1b)
$\hat{a}_3(\hat{A}_3)$	unit normal to the undeformed (deformed) reference surface $\theta^3 = 0$
$a_{\alpha\beta}(A_{\alpha\beta})$	component of metric tensor of undeformed (deformed) reference surface; c.f. equations (3), (7)
$a(A)$	determinant $\lvert a_{\alpha\beta}\rvert(\lvert A_{\alpha\beta}\rvert)$
$\Gamma^\gamma_{\alpha\beta}(^*\Gamma^\gamma_{\alpha\beta})$	Christoffel symbols of the undeformed (deformed) surface coordinates; c.f. equations (4), (9)
$\varepsilon_{\alpha\beta}(^*\varepsilon_{\alpha\beta})$	$e_{\alpha\beta}\sqrt{a}(e_{\alpha\beta}\sqrt{A})$ where $e_{\alpha\beta}$ denotes the permutation symbol
$b_{\alpha\beta}(B_{\alpha\beta})$	coefficients in the second fundamental form of the reference surface; c.f. equations (5), (6), (8)
$\vec{c}_i$(or $\vec{c}^i$)	component of triad similar to $\vec{a}_i$ but rotated; c.f. Fig. 3
${}_0\gamma_{\alpha\beta}$	component of strain at the reference surface; c.f. equation (14)
$\varkappa_{\alpha\beta}$	change-of-curvature of the reference surface; c.f. equation (15)
γ_α	transverse shear strain; c.f. equation (18)
$\eta_{\alpha\beta}$	flexural strain-measure; c.f. equation (20)
$\vec{w}$	relative displacement of reference surface; c.f. equation (22)

$\vec{\bar{\omega}}$	relative rotation of reference surface; c.f. equation (29), (33)
$\omega_{i\alpha}$	components of relative rotation; c.f. equations (24), (26), (29)
E^{ijkl}	stiffness tensor; ref. [11]
$\vec{C}^{\alpha\beta\gamma\eta}$	surface-stiffness tensor; c.f. equations (37), (38)
$r^{i}_{\cdot j}({}_{Q}r^{i}_{\cdot j})$	tensor defining rotation of principal lines of strain (at reference surface); c.f. equations (44), (46)
$\vec{\beta}$	small rotation of normal produced by transverse shear; c.f. equations (52), (53), (54), (57)
${}_{Q}\tilde{r}^{i}_{\cdot j}$	tensor defining finite rotation of the reference surface; c.f. equation (55)
${}_{Q}\bar{r}^{i}_{\cdot 3}$	component of tensor defining rotation of normal, c.f. equation (58)
$\vec{\phi}$	small relative rotation of normal; c.f. equation (59), (60)
$\vec{P}$	position vector of base point
$\vec{p}({}_{Q}\vec{p})$	position of generic point (on reference surface) relative to base point
$\vec{\Omega}$	angular velocity of convecting triad $\vec{c}_i$; c.f. equation (70)
T	kinetic energy of finite element
U	strain energy of finite element
$\vec{V}$	displacement vector
$\delta\psi$	incremental rotation of normal; c.f. equation (114c)
$\delta\alpha$	incremental rotation of reference surface; c.f. equations (114b, c)
q_N	generalized coordinate
p_N	total generalized force upon element; $p_N = t_N + f_N$
f_N	generalized edge force at location N
t_N	generalized external force at location N
f^{i}_{N}	component of actual force at location N
$m^{\alpha N}$	component of actual couple at location N
$r^{j}_{\cdot i}(PQ)$	component $r^{j}_{\cdot i}$ of tensor evaluated at point P but shifted to the base of point Q; see Appendix
$C^{i}_{\cdot j}(OQ)$	shifter; c.f. equation (126)
$\vec{f}_{N_A}$	force at edge point N of element A

Notations not listed are defined in the text.

PART I: THEORY

Introduction

THE deformation of a continuous medium in the neighborhood of a particle is described by a linear displacement field [1]. The description is accurate in a neighborhood of the particle and approximate in a small finite region containing the particle. If a body is subdivided into small finite regions, then a continuous displacement field can be approximated by a field which is piecewise linear in each coordinate, linear in each subregion. This notion has been employed to approximate a continuous body by a discrete system. Numerous references are given by Argyris [2]. The approximation within a finite region can be accomplished by the energy method of Ritz. If the entire displacement field is continuous across the interfaces of subregions, then it can approach an arbitrary continuous field in the limit as the size of the subregions diminishes.

The motion of a continuous medium in the neighborhood of a particle can be decomposed into a rigid body motion followed by a deformation (or vice-versa) [3]. If finite-elements are small enough to provide a valid approximation of the continuous body, then the motion of each finite-element can be similarly decomposed. In particular, the motion of a finite-element of a flexible shell can be conceived as a *finite* rigid motion followed by a *small* relative motion. The latter motion is not involved with geometrical nonlinearities and the former is not involved in the constitutive equations for the element. If the material is Hookean, the constitutive equations are linear yet apply to circumstances with finite rotations. Here, the decomposition is applied to shells and, more specifically, to finite-elements with an important consequence: Any existing linear approximation for finite-elements can be employed to treat situations involving finite rotations. An example demonstrates the applicability of the decomposition.

Finite-element representations [4, 5, 6] of plates and shells have adhered to the Kirchhoff hypothesis, assuming that a material line, initially normal to a reference surface, remains normal during deformation. Continuity can be achieved at the interfaces of quadrilateral elements if the displacement-field is approximated by a sixth-degree polynomial [6]. However, the simplicity of the piecewise linear representation is lost, the individual element has 24 degrees-of-freedom and the derivation of the stiffness matrix is a formidable task in itself. Alternative approaches are given by Clough and Tocher [4] and Fraeijs de Veubke [5]. Both achieve continuity at interelement boundaries by procedures in which the elements are subdivided and subsequently reassembled. The former [4] describes a triangular (HCT) plate-element with 9 degrees-of-freedom for bending (additional coordinates are needed for stretching) and the latter [5] presents a quadrilateral plate-element with 16 degrees-of-freedom for bending.

Much of the difficulty in the application of finite-element formulations can be attributed to insistence upon the Kirchhoff hypothesis. Accordingly, we present here a theory which relaxes the Kirchhoff hypothesis and admits transverse shear strain. The theory permits far simpler approximations of the displacement field including the piecewise linear approximation.

The shear-deformation theory serves primarily to achieve continuity at inter-element edges. Further simplifications result from constraints analogous to the Kirchhoff hypothesis.

The simplest approximating polynomials enforce severe transverse shear strains which, in turn, produce excessively stiff elements. This difficulty can be eliminated, after introducing Kirchhoff-type constraints, by suppressing the strain energy caused by transverse shear. A numerical example demonstrates the merit of the resulting theory.

The presentation is divided in two parts: Part I contains the kinematical foundations, specifically, the decomposition of rotation and strain, and the generalization of the Kirchhoff theory. It is essentially a theory of shells, but a theory which is basic to our treatment of finite rotations and our utilization of simple finite-element approximations.

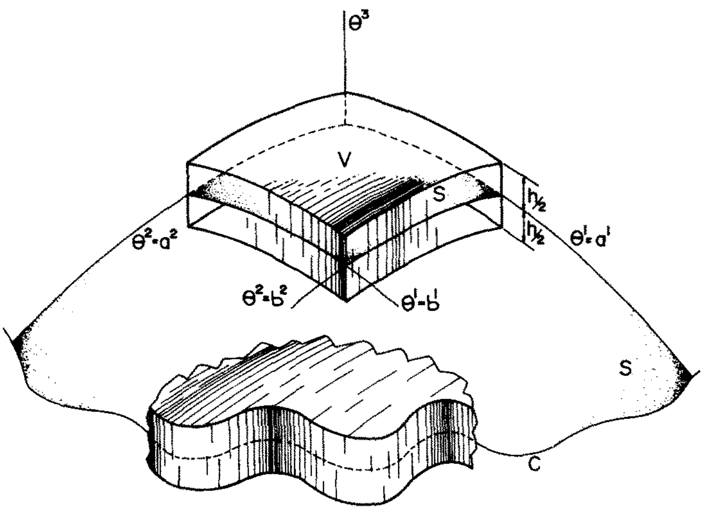

FIG. 1

Part II describes the application of the theory to an arbitrary shell and presents a few illustrative examples.

The finite element

Let θ^α denote arbitrary coordinates of a reference surface $\mathcal{S}$ in Fig. 1 and θ^3 the distance along the normal to $\mathcal{S}$. The undeformed shell is the region bounded by the surfaces $\theta^3 = \pm h/2$ and an edge defined by a curve C on $\mathcal{S}$ and the normals to the surface along C.

A finite element of the undeformed shell is the region V bounded by the surfaces $\theta^3 = \pm h/2$ and edges formed along the nearby coordinate lines $\theta^\alpha = a^\alpha$, b^α which enclose the finite-element surface S.

Motion of an element

The motion of an element can be conceived as a rigid-body motion followed by a deformation. The hexahedral element V of Fig. 2 is rigidly transported to V'' and then

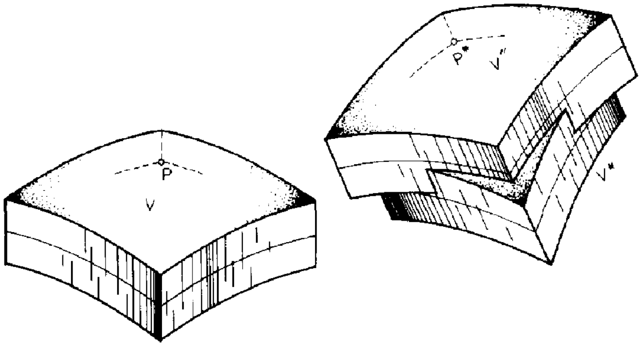

FIG. 2

deformed to V^*. A quadrilateral element S of the reference surface is rigidly transported to S'' and then deformed to S^* as depicted in Fig. 3.

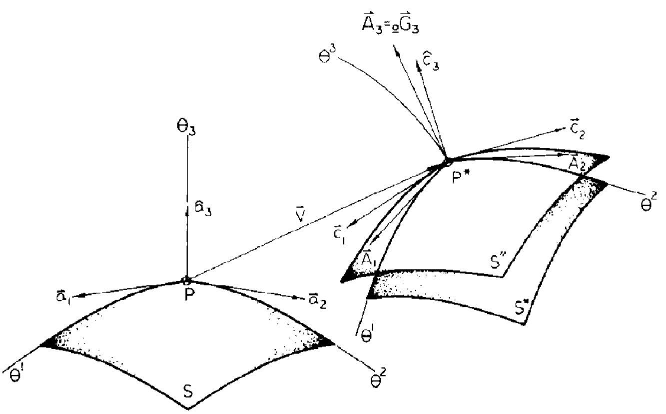

FIG. 3

Kinematics of the surface element

Let ${}_0\vec{r}$ and ${}_0\vec{R}$ denote the position vector of a particle before and after deformation. A tangent vector to the θ^α coordinate line before deformation is

$$\vec{a}_\alpha \equiv \frac{\partial_0 \vec{r}}{\partial \theta^\alpha} \tag{1a}$$

and the tangent vector to the deformed line is

$$\vec{A}_\alpha \equiv \frac{\partial_0 \vec{R}}{\partial \theta^\alpha}. \tag{1b}$$

The base vectors $\vec{a}_i$ of Fig. 3 are tangent to the undeformed coordinate lines and the vectors $\vec{A}_i$ are tangent to the deformed coordinate lines. A rigid motion carries the particle P to P^*, the surface S to S'' and the triad $\vec{a}_i(P)$ to the similar triad $\vec{c}_i(P^*)$. A subsequent deformation carries S'' to S^* and deforms the triad $\vec{c}_i(P^*)$ to $\vec{A}_i(P^*)$.

A fundamental theorem [7] asserts that the motion at a particle can be decomposed into a translation, a rigid rotation of the principal lines (of strain) and a subsequent stretching of the principal lines. The rotation in question is then the rotation of principal lines. In the analysis of a finite element of a *shell* it is more meaningful to employ a rotation which rigidly transports the reference surface S to a position S'' tangent to the deformed surface S^* at a reference particle P^*. The rigid-body motion is completely defined and the orientation of the triad $\vec{c}_i(P^*)$ is fixed if we require that

$$\vec{A}_\alpha \cdot \vec{c}_\beta = \vec{c}_\beta \cdot \vec{A}_\alpha \equiv \phi_{\alpha\beta} \tag{2a,b}$$

$$\vec{A}_\alpha \cdot \vec{c}_3 = 0 \tag{2c}$$

at point P^*. In the sense of small strains, the motion which carries S'' to S^* involves no gross rotation. However, if the shell suffers transverse shear strain, then the motion involves a rotation of principal lines. The computation of this rotation is described later.

The surface S'' and vectors $\vec{c}_i$ differ from S and $\vec{a}_i$ by a rigid-body motion. The differential geometry of S'' and S is identical. It follows that a covariant component of the metric tensor [8] is

$$\vec{c}_\alpha \cdot \vec{c}_\beta = \vec{a}_\alpha \cdot \vec{a}_\beta \equiv a_{\alpha\beta}. \tag{3a, b}$$

The reciprocal vectors $\vec{a}^i$ and $\vec{c}^i$ form similar triads but also differ by a rigid-body rotation. It follows too that the Christoffel symbols of surface S and S'' are identical:

$$\Gamma^\mu_{\alpha\beta} \equiv \vec{a}^\mu \cdot \vec{a}_{\alpha,\beta} = \vec{c}^\mu \cdot \vec{c}_{\alpha,\beta} \tag{4a, b}$$

etc. A covariant component of the second fundamental tensor [8] is

$$b_{\alpha\beta} \equiv \hat{a}_3 \cdot \vec{a}_{\alpha,\beta} = \hat{c}_3 \cdot \vec{c}_{\alpha,\beta} \tag{5a, b}$$

$$= -\vec{a}_\alpha \cdot \hat{a}_{3,\beta} = -\vec{c}_\alpha \cdot \hat{c}_{3,\beta}. \tag{6a, b}$$

The first and second fundamental tensors of the deformed surface S^* are similarly defined, i.e.

$$A_{\alpha\beta} \equiv \vec{A}_\alpha \cdot \vec{A}_\beta, \qquad B_{\alpha\beta} \equiv \hat{A}_3 \cdot \vec{A}_{\alpha,\beta} = -\vec{A}_\alpha \cdot \hat{A}_{3,\beta} \tag{7), (8a, b}$$

etc. where $\hat{A}_3 \neq \vec{A}_3$, but denotes the unit normal to S^*. We denote the Christoffel symbols of the deformed surface by

$$^*\Gamma^{\mu}_{\alpha\beta} = \vec{A}^{\mu} \cdot \vec{A}_{\alpha,\beta}. \tag{9}$$

etc.

Kinematics of the element

The position vector of an arbitrary particle of the undeformed element is

$$\vec{r}(\theta^1, \theta^2, \theta^3) = {}_0\vec{r}(\theta^1, \theta^2) + \theta^3 \hat{a}_3. \tag{10}$$

We now assume that the position of a particle of the deformed body is a linear function of θ^3:

$$\vec{R}(\theta^1, \theta^2, \theta^3) = {}_0\vec{R}(\theta^1, \theta^2) + \theta^3 {}_0\vec{G}_3(\theta^1, \theta^2). \tag{11}$$

In words, the approximation (11) asserts that a material-line, initially normal to S, remains straight. We next label the components of ${}_0\vec{G}_3$ as follows:

$${}_0\vec{G}_3 \equiv (1+\varepsilon)\hat{A}_3 + 2\gamma^{\alpha}\vec{A}_{\alpha}. \tag{12}$$

From (11) and (12), it follows that

$$\vec{R}_{,\alpha} = [\delta^{\mu}_{\alpha} + \theta^3(2\gamma^{\mu}\overset{*}{|}_{\alpha} - B^{\mu}_{\alpha} - \varepsilon B^{\mu}_{\alpha})]\vec{A}_{\mu} + \theta^3(2\gamma^{\mu}B_{\mu\alpha} + \varepsilon_{,\alpha})\hat{A}_3 \tag{13a}$$

$$\vec{R}_{,3} = {}_0\vec{G}_3 \tag{13b}$$

where $(\overset{*}{|})$ signifies covariant differentiation with respect to the deformed surface.

We define two tensors which characterize the deformation of the surface at a point:

$${}_0\gamma_{\alpha\beta} \equiv \tfrac{1}{2}(A_{\alpha\beta} - a_{\alpha\beta}) \tag{14}$$

$$\varkappa_{\alpha\beta} \equiv \tfrac{1}{2}(B_{\alpha\beta} - b_{\alpha\beta}). \tag{15}$$

Throughout the subsequent development the variables γ_{α}, ${}_0\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$ are supposedly small, i.e. the corresponding physical components are small compared to unity.

For brevity, let

$$\vec{r}_{,i} = \vec{g}_i, \qquad \vec{R}_{,i} = \vec{G}_i. \tag{16a, b}$$

The components of Green's [9] strain tensor are determined from (13) with the notations of (7), (8), (14), (15) and (16). Neglecting products of small quantities γ_{α}, ${}_0\gamma_{\alpha\beta}$, $\varkappa_{\alpha\beta}$, we obtain

$$\gamma_{33} = \tfrac{1}{2}(\vec{G}_3 \cdot \vec{G}_3 - 1) \doteq \varepsilon \tag{17a, b}$$

$$\gamma_{3\alpha} \equiv \tfrac{1}{2}\vec{G}_3 \cdot \vec{G}_{\alpha} \doteq \gamma_{\alpha} + \frac{\theta^3}{2}\varepsilon_{,\alpha} \tag{18a, b}$$

$$\gamma_{\alpha\beta} \equiv \tfrac{1}{2}(\vec{G}_{\alpha} \cdot \vec{G}_{\beta} - \vec{g}_{\alpha} \cdot \vec{g}_{\beta}) \doteq {}_0\gamma_{\alpha\beta} + \theta^3\eta_{\alpha\beta} \tag{19a, b}$$

wherein

$$\eta_{\alpha\beta} \equiv \gamma_{\alpha}|_{\beta} + \gamma_{\beta}|_{\alpha} - \varkappa_{\alpha\beta}. \tag{20}$$

Here we have replaced the covariant differentiation $(|^*)$ with respect to the deformed surface by differentiation with respect to the undeformed surface $(|)$, assuming that derivatives of the strain components are also small. In addition, if the transverse strain ε varies gradually we can neglect the underlined term of (18b). This term is neglected in the sequel.

The deformation at a particle of the shell is characterized by nine variables: ε, γ_α, ${}_0\gamma_{\alpha\beta}$, and $\varkappa_{\alpha\beta}$. Our theory gives the strain distribution:

$$\gamma_{33} = \varepsilon, \qquad \gamma_{3\alpha} = \gamma_\alpha \tag{21a, b}$$

$$\gamma_{\alpha\beta} = {}_0\gamma_{\alpha\beta} + \theta^3 \eta_{\alpha\beta}. \tag{21c}$$

Strain and relative displacement of an element

Let $\vec{w}(\theta^1, \theta^2)$ denote the *small relative* displacement which carries S'' of Fig. 3 to S^*.

$$\vec{w} \equiv w^\lambda \vec{c}_\lambda + w^3 \hat{c}_3 \tag{22}$$

$$\vec{w}_{,\alpha} \equiv \vec{A}_\alpha - \vec{c}_\alpha \tag{23a}$$

$$= (w^\lambda|_\alpha - w^3 b^\lambda_\alpha)\vec{c}_\lambda + (w^3_{,\alpha} + w^\lambda b_{\lambda\alpha})\hat{c}_3. \tag{23b}$$

The normal component is:

$$\omega_{3\alpha} \equiv \hat{c}_3 \cdot \vec{w}_{,\alpha} = w^3_{,\alpha} + w^\lambda b_{\lambda\alpha}. \tag{24a, b}$$

The symmetric and skew-symmetric parts of the tangential components are:

$$\phi_{\alpha\beta} \equiv \tfrac{1}{2}(\vec{c}_\alpha \cdot \vec{w}_{,\beta} + \vec{c}_\beta \cdot \vec{w}_{,\alpha}) \tag{25a}$$

$$= \tfrac{1}{2}(w_\alpha|_\beta + w_\beta|_\alpha - 2w^3 b_{\alpha\beta}) \tag{25b}$$

$$\omega_{\alpha\beta} \equiv \tfrac{1}{2}(\vec{c}_\alpha \cdot \vec{w}_{,\beta} - \vec{c}_\beta \cdot \vec{w}_{,\alpha}) \tag{26a}$$

$$= \tfrac{1}{2}(w_\alpha|_\beta - w_\beta|_\alpha). \tag{26b}$$

From (23), (24), (25), and (26) it follows that

$$\vec{A}_\alpha = \vec{c}_\alpha + (\phi_{\beta\alpha} + \omega_{\beta\alpha})\vec{c}^\beta + \omega_{3\alpha}\hat{c}_3. \tag{27}$$

Since the motion from S'' to S^* is characterized by small strain and small relative rotations, the surface tensors $\phi_{\alpha\beta}$, $\omega_{\alpha\beta}$ and $\omega_{3\alpha}$ can be identified with the surface strain and rotation as follows:

$${}_0\gamma_{\alpha\beta} \doteq \phi_{\alpha\beta} \tag{28}$$

$$\vec{\omega} \equiv \omega^i \vec{c}_i \tag{29a}$$

$$\equiv \varepsilon^{\alpha\mu}(\omega_{3\mu}\vec{c}_\alpha + \tfrac{1}{2}\omega_{\mu\alpha}\hat{c}_3) \tag{29b}$$

wherein $\varepsilon^{\alpha\beta}$ is a surface tensor related to the permutation symbol $e^{\alpha\beta}$ via

$$\varepsilon^{\alpha\beta} = e^{\alpha\beta}/\sqrt{a} \tag{30}$$

$$\sqrt{a} = \hat{a}_3 \cdot (\vec{a}_1 \times \vec{a}_2). \tag{31}$$

If the reference triad $\vec{c}_i$ is chosen in accordance with (2), then according to (23), (24), (26), and (29) $\omega^i = 0$, $\omega_{i\alpha} = 0$ *at* the reference point P^*.

With the definition (29), equation (27) has the alternative form:

$$\vec{A}_\alpha = \vec{c}_\alpha + \phi_{\beta\alpha}\vec{c}^\beta + \vec{\omega}\times\vec{c}_\alpha. \tag{32}$$

We note that (32) is exact, but that the tensor $\phi_{\alpha\beta}$ and vector ω_i can be identified with the surface strain and rotation only if the latter are small throughout the region in question.

In the notation of (24), (26), and (29) the normal to the deformed surface is given by

$$\hat{A}_3 = \hat{c}_3 + \vec{\omega}\times\hat{c}_3 \tag{33a}$$

$$= \hat{c}_3 - (\hat{c}_3\cdot\vec{w}_{,\beta})\vec{c}^\beta. \tag{33b}$$

From (12), (28), (32), (33) and the approximations of small strain and small relative rotation, it follows that

$$\vec{A}_\alpha \equiv {}_0\vec{G}_\alpha = \vec{c}_\alpha + {}_0\gamma^\beta_\alpha\vec{c}_\beta + \vec{\omega}\times\vec{c}_\alpha \tag{34a}$$

$$\vec{G}_3 = {}_0\vec{G}_3 = 2\gamma^\alpha\vec{c}_\alpha + (1+\varepsilon)\hat{c}_3 + \vec{\omega}\times\hat{c}_3. \tag{34b}$$

In accordance with (5), (8), (15), (23), (33), (24), (25), and (26),

$$\varkappa_{\alpha\beta} = \omega_{3\alpha}|_\beta + (\phi_{\mu\alpha}+\omega_{\mu\alpha})b^\mu_\beta. \tag{35a}$$

Finally, we note that the term $\phi_{\mu\alpha}b^\mu_\beta$ can be neglected [10] and that the change-of-curvature tensor must be symmetric. Consequently, equation (35a) can be replaced by

$$\varkappa_{\alpha\beta} = \tfrac{1}{2}(\omega_{3\alpha}|_\beta + \omega_{3\beta}|_\alpha + \omega_{\mu\alpha}b^\mu_\beta + \omega_{\mu\beta}b^\mu_\alpha). \tag{35b}$$

Strain energy of an element

Assuming that the transverse normal stress is negligible and that the material is linearly elastic and elastically symmetric with respect to the middle surface, we take the strain energy in the form [11]:

$$\Phi = \tfrac{1}{2}C^{\alpha\beta\gamma\eta}\gamma_{\alpha\beta}\gamma_{\gamma\eta} + 2E^{\alpha3\beta3}\gamma_{\alpha3}\gamma_{\beta3} \tag{36}$$

where

$$C^{\alpha\beta\gamma\eta} = E^{\alpha\beta\gamma\eta} - \frac{E^{33\alpha\beta}E^{33\gamma\eta}}{E^{3333}} \tag{37}$$

and E^{ijkl} are the elastic coefficients of the three-dimensional isotropic or aelotropic body.

For the thin shell, it is consistent to evaluate the components E^{ijkl} at the middle surface, i.e.

$$E^{ijkl} \doteq E^{ijkl}(\theta^1,\theta^2,0) \equiv \bar{E}^{ijkl} \tag{38a, b}$$

$$C^{\alpha\beta\gamma\eta} \doteq C^{\alpha\beta\gamma\eta}(\theta^1,\theta^2,0) \equiv \bar{C}^{\alpha\beta\gamma\eta}. \tag{38c, d}$$

Substituting (21b, c) and (38) into (36) and integrating through the thickness, we obtain

$$\phi \equiv \int_{-h/2}^{h/2}\Phi\sqrt{\frac{g}{a}}\,d\theta^3 \tag{39a}$$

$$\doteq \frac{h}{2}\bar{C}^{\alpha\beta\gamma\eta}\left({}_0\gamma_{\alpha\beta}\,{}_0\gamma_{\gamma\eta} + \frac{h^2}{12}\eta_{\alpha\beta}\eta_{\gamma\eta}\right) + 2h\bar{E}^{\alpha3\beta3}\gamma_\alpha\gamma_\beta. \tag{39b}$$

The strain energy of a finite element is obtained by integrating (39) over the finite surface S which defines the element:

$$U \equiv \int_{a^1}^{b^1} \int_{a^2}^{b^2} \phi\sqrt{a}\, d\theta^1\, d\theta^2. \tag{40}$$

Finite rotation and small relative rotations

The Appendix contains an analysis which decomposes the motion of a neighborhood into a finite rotation followed by a deformation. The rotation of the principal lines of strain is represented by the tensor $r^{i}_{\cdot j}$ of (82) and the deformation by the tensor $({}_{\frac{1}{2}}G^j_i)$ of (83). In the case of small strain, the rotation tensor is expressed by the approximation (85), the deformation by the approximation of (84) and the motion which successively rotates and deforms the base triad $\vec{g}_i$ to $\vec{g}'_i$ to $\vec{G}_i$ is given by (82) and (86).

Let us examine the motion at the reference surface S, where we set

$$\vec{g}_i(\theta^1, \theta^2, 0) = \vec{a}_i, \qquad \vec{g}'_i(\theta^1, \theta^2, 0) = \vec{a}'_i \tag{41),(42}$$

$$\vec{G}_i(\theta^1, \theta^2, 0) = {}_0\vec{G}_i \tag{43}$$

$$r^{i}_{\cdot j}(\theta^1, \theta^2, 0) = {}_0r^{i}_{\cdot j} \tag{44}$$

$$\gamma^\alpha_\beta(\theta^1, \theta^2, 0) = {}_0\gamma^\alpha_\beta, \qquad \gamma^3_3(\theta^1, \theta^2, 0) = \varepsilon, \qquad \gamma^3_\alpha(\theta^1, \theta^2, 0) = \gamma_\alpha. \tag{45a, b, c}$$

In these notations, equation (82) defines a rigid rotation at the reference surface S; specifically,

$$\vec{a}'_i = {}_0r^{j}_{\cdot i}\vec{a}_j. \tag{46}$$

With small strain, the tensor ${}_0r^i_j$ is given by (85); in particular,

$${}_0r^{i}_{\cdot\alpha} = \vec{a}^i \cdot [(\delta^\beta_\alpha - {}_0\gamma^\beta_\alpha){}_0\vec{G}_\beta - \gamma_\alpha\, {}_0\vec{G}_3]. \tag{47}$$

According to (86) a tangent to the deformed surface S^* is

$${}_0\vec{G}_\alpha = (\delta^\beta_\alpha + {}_0\gamma^\beta_\alpha)\vec{a}'_\beta + \gamma_\alpha \hat{a}'_3 \tag{48}$$

and the tangent to the deformed θ^3 line is

$${}_0\vec{G}_3 = (1+\varepsilon)\hat{a}'_3 + \gamma^\alpha \vec{a}'_\alpha. \tag{49}$$

Upon examining (47), we note that the rotation tensor depends on the transverse shear strain γ_α and, upon examining (48), we observe that the rotated vector $\vec{a}'_\alpha$ is not tangent to the deformed surface S^*.

In the analysis of shells, it is appropriate to deal with the motion of a reference surface. The rotation of the reference surface does not involve the transverse shear strain. Accordingly, we conceive a decomposition in which the base triad $\hat{a}_i$ at point P is, first, rigidly transported and rotated to a similar triad $\vec{c}'_i$ at point P^* such that the vector $\vec{c}'_\alpha$ is tangent to the deformed surface S^*. Then the triad $\vec{c}'_i$ complies with Equations (2a) and (2c) which imply that the relative rotation $\vec{\omega}$ of (34) vanishes. Consequently, the latter equations reduce to

$$\vec{A}_\alpha = {}_0\vec{G}_\alpha = (\delta^\beta_\alpha + {}_0\gamma^\beta_\alpha)\vec{c}'_\beta \tag{50}$$

$$\vec{G}_3 = {}_0\vec{G}_3 = 2\gamma^\alpha \vec{c}'_\alpha + (1+\varepsilon)\vec{c}'_3. \tag{51}$$

By comparing (48) and (49) with (50) and (51), we conclude that the triads $\vec{a}_i'$ and $\vec{c}_i'$ differ by a *small* rotation; specifically,

$$\vec{c}_i' = \vec{a}_i' + \vec{\beta}\times\vec{a}_i' \tag{52}$$

wherein

$$\vec{\beta} \equiv \varepsilon^{\gamma\alpha}\gamma_\alpha\vec{a}_\gamma' \doteq \varepsilon^{\gamma\alpha}\gamma_\alpha\vec{c}_\gamma'. \tag{53a, b}$$

The inverse of (52) is

$$\vec{a}_i' = \vec{c}_i' - \vec{\beta}\times\vec{c}_i'. \tag{54}$$

The relative orientation of $\vec{a}_i'$ and $\vec{c}_i'$ is shown at the point P^* of Fig. 4.

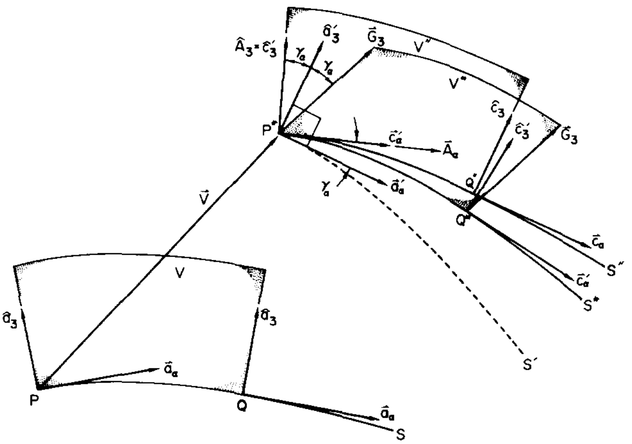

FIG. 4

The rigid motion which carried $\vec{a}_i$ to $\vec{c}_i'$ is expressed by an orthogonal transformation:

$$\vec{c}_i' = {}_{\underline{0}}\tilde{r}^{j}_{\cdot i}\vec{a}_j. \tag{55}$$

By inserting (46) in the right side of (52) and equating (52) with (55) we obtain

$${}_{\underline{0}}\tilde{r}^{j}_{\cdot i} = {}_{\underline{0}}r^{j}_{\cdot i} + \gamma_\alpha\delta^\alpha_i\,{}_{\underline{0}}r^{j}_{\cdot 3} - \gamma^\alpha\delta^3_i\,{}_{\underline{0}}r^{j}_{\cdot\alpha} \tag{56a}$$

Conversely,

$${}_{\underline{0}}r^{j}_{\cdot i} = {}_{\underline{0}}\tilde{r}^{j}_{\cdot i} - \gamma_\alpha\delta^\alpha_i\,{}_{\underline{0}}\tilde{r}^{j}_{\cdot 3} + \gamma^\alpha\delta^3_i\,{}_{\underline{0}}\tilde{r}^{j}_{\cdot\alpha}. \tag{56b}$$

We observe that the motion which carries $\hat{a}_3$ to $\hat{c}_3'$ differs with that which carries $\hat{a}_3$ to $\vec{G}_3$. The deformed normal $\vec{G}_3$ results from an additional rotation $(-2\vec{\beta})$ caused by transverse shear and an extensional strain (ε). In accordance with (51) the *unit* vector $\hat{G}_3$ tangent to the deformed θ^3 line is

$$\hat{G}_3 = \hat{c}_3' - 2\vec{\beta}\times\hat{c}_3'. \tag{57}$$

The transformation which carries the initial vector $\hat{a}_3$ to the vector $\hat{G}_3$ is obtained via (57) and (55) as follows:

$$\hat{G}_3 = ({}_Q\tilde{r}^{j}_{\cdot 3} + 2\gamma^{\eta}\, {}_Q\tilde{r}^{j}_{\cdot \eta})\vec{a}_j \tag{58a}$$

$$\equiv {}_Q\tilde{r}^{j}_{\cdot 3}\vec{a}_j. \tag{58b}$$

For our purposes, we may conceive the motion of a neighborhood V of P decomposed into a rigid-body motion which carries V to V'' and a deformation which deforms V'' to V^*. The rigid motion transports the particle at P to its terminal position P^* and the reference surface S to S'' tangent to the deformed surface S^*. The differential geometry at a generic point Q'' of S'' and V'' is identical to that at the same particle at Q of S and V. The base vector $\hat{c}_i$, tangent to the displaced θ^i coordinate-line, is similar to $\vec{a}_i$ but rotated so that

$$\vec{c}_i(P^*) = \vec{c}_i'(P^*).$$

The surfaces S, S'', and S^* and vectors $\vec{a}_i$, $\vec{c}_i'$, $\vec{c}_i$ are depicted in Fig. 4.

From the latter viewpoint, the deformation which deforms V'' to V^* and S'' to S^* is characterized by small relative rotations. Consequently, the deformation rotates and deforms the triad $\vec{c}_i$ to ${}_Q\vec{G}_i$ in accordance with (34a) and (34b); the latter can be rewritten in the form:

$$\vec{G}_3 = {}_Q\vec{G}_3 = (1+\varepsilon)\hat{c}_3 + \vec{\phi}\times\hat{c}_3 \tag{59}$$

where $\vec{\phi}$ is the rotation which carries $\hat{c}_3$ to $\hat{G}_3$:

$$\phi^{\beta} = \varepsilon^{\alpha\beta}(2\gamma_{\alpha} + \omega_{\alpha 3}), \qquad 2\gamma_{\alpha} = \varepsilon_{\alpha\beta}\phi^{\beta} - \omega_{\alpha 3}. \tag{60a, b}$$

Equating the right side of (50) with (34a) and (51) with (59), and invoking the small-strain approximation, we obtain

$$\vec{c}_i' = \vec{c}_i + \vec{\varpi}\times\vec{c}_i \tag{61a}$$

$$= (\delta_i^j + \omega^{j}_{\cdot i})\hat{c}_j. \tag{61b}$$

By equating (55) and (61) we obtain

$${}_Q\tilde{r}^{j}_{\cdot i} = \vec{a}^j\cdot\vec{c}_P(\delta_i^P + \omega^{P}_{\cdot i}). \tag{62}$$

Now the vector $\vec{c}_p$ is a base vector at a generic point Q'' of S'', obtained by a rotation of $\vec{a}_p$ at the corresponding point Q of S. However, the rotation is that of the reference point P. To effect the rotation, the tensor ${}_Q\tilde{r}^{j}_{\cdot i}$ (P) must be shifted to the base of Q as described in the Appendix. The shifted component is [see (108)]

$${}_Q\tilde{r}^{j}_{\cdot i}(PQ) = [\vec{a}^p(P)\cdot\vec{a}_i(Q)][\vec{a}_m(P)\cdot\vec{a}^j(Q)]\,{}_Q\tilde{r}^{m}_{\cdot p}. \tag{63}$$

With this notation, (62) takes the form

$${}_Q\tilde{r}^{j}_{\cdot i}(Q) = {}_Q\tilde{r}^{j}_{\cdot p}(PQ)[\delta_i^p + \omega^{p}_{\cdot i}(Q)]. \tag{64}$$

Equations (62) and (64) are particularly significant in analyses of finite rotations by the method of finite elements, for they express the rotation at a point Q near P in terms of the rotation at P and the small relative rotation $\vec{\varpi}$.

The rotation of the normal is given by the tensor ${}_0\bar{r}^{i}_{\cdot 3}$ of (58). The rotation of $\vec{G}_3$ relative to the local base $\vec{c}_i$ of the reference surface S'' is $\vec{\omega}-2\vec{\beta}$. Accordingly, the counterpart of (64) is

$${}_0\bar{r}^{j}_{\cdot 3}(Q) = {}_0\bar{r}^{j}_{\cdot p}(PQ)[\delta^p_3+\omega^p_{\cdot 3}(Q)+2\delta^p_\eta\gamma^\eta(Q)]. \tag{65}$$

The tensor ${}_0\bar{r}^{i}_{\cdot 3}$ must be continuous; in particular, it must be continuous along interelement boundaries.

If the transverse shear strains vanish, i.e. $\gamma_\alpha \equiv 0$, *then*

$${}_0r^{i}_{\cdot j} = {}_0\tilde{r}^{i}_{\cdot j} \tag{66}$$

and

$${}_0\bar{r}^{i}_{\cdot 3} = {}_0\tilde{r}^{i}_{\cdot 3} = {}_0r^{i}_{\cdot 3}. \tag{67}$$

Velocity and kinetic energy

In our analysis of the motion of a finite element we require an expression for the velocity of an arbitrary particle at Q^*. To this end, let $\vec{P}$ denote the position vector of the particle at P^* and $\vec{p}$ the position vector of Q^* *relative* to P^*. The velocity of the particle at Q^* is

$$\dot{\vec{V}} = \dot{\vec{P}}+\dot{\vec{p}}. \tag{68}$$

The vector $\dot{\vec{P}}$ is the velocity of the particle at P^*. The velocity $\dot{\vec{p}}$ can be decomposed into the contribution from the rigid-body motion and the deformation.

If $\vec{\Omega}$ denotes the angular velocity of triad $\vec{c}_i$, then

$$\dot{\vec{p}} = \vec{\Omega}\times\vec{p}+\dot{\vec{w}}+\theta^3(\dot{\vec{\phi}}\times\hat{c}_3+\dot{\varepsilon}\hat{c}_3) \tag{69}$$

wherein $\vec{w}$, $\vec{\phi}$ and ε are the relative displacement of (22), the relative rotation of (59) and the extensional strain of (21a).

The angular velocity $\vec{\Omega}$ is expressed in terms of the rotation tensor ${}_0\tilde{r}^{j}_{\cdot i}$ of (63) as follows:

$$\dot{\vec{c}}_i = \vec{\Omega}\times\vec{c}_i \tag{70a}$$

$$= {}_0\dot{\tilde{r}}^{j}_{\cdot i}\vec{a}_j. \tag{70b}$$

Therefore,

$${}_0\dot{\tilde{r}}^{j}_{\cdot i} = \vec{a}^j\cdot(\vec{\Omega}\times\vec{c}_i) \tag{71a}$$

$$= {}_0\tilde{r}^{p}_{\cdot i}\vec{a}^j\cdot(\vec{\Omega}\times\vec{a}_p) \tag{71b}$$

$$= {}_0\tilde{r}^{p}_{\cdot i}\Omega^l a^{jm}\,{}_0\varepsilon_{pml}. \tag{71c}$$

Conversely,

$$2\Omega_p = {}_0\dot{\tilde{r}}^{j}_{\cdot i}\,{}_0\tilde{r}^{ki}\,{}_0\varepsilon_{kjp}. \tag{71d}$$

All components are referred to the initial base-system $\vec{a}_i$.

If the rotations are moderate, then in the manner of (88):

$${}_0\tilde{r}_{ji} = a_{ji}+\theta^k\,{}_0\varepsilon_{kij} \tag{72a}$$

$${}_0\dot{\tilde{r}}_{ji} = \dot{\theta}^k\,{}_0\varepsilon_{kij} \tag{72b}$$

$$\Omega_p \doteq \dot{\theta}_p. \tag{72c}$$

The kinetic energy of the finite element is the integral

$$T = \frac{1}{2}\int\int\int_V \rho\dot{\vec{V}}\cdot\dot{\vec{V}}\,\mathrm{d}v \tag{73}$$

where ρ denotes the mass density of the undeformed medium.

If the shell is thin, the volume element is approximately

$$\mathrm{d}v \doteq \sqrt{a}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3. \tag{74}$$

To further simplify (73), we neglect the extension rate $\dot{\varepsilon}$ in (69), assume that $\rho = \rho(\theta^1, \theta^2)$ and set

$$\vec{p} = {}_0\vec{p}(\theta^1, \theta^2) + \theta^3\hat{c}_3. \tag{75}$$

With the approximations (74) and (75), (73) assumes the form

$$\begin{aligned} T = &\left[\frac{M}{2}\dot{\vec{P}}\cdot\dot{\vec{P}} + h(\dot{\vec{P}}\times\vec{\Omega})\cdot\int\int_S {}_0\vec{p}\rho\,\mathrm{d}S\right. \\ &\left. + \frac{h}{2}\int\int(\vec{\Omega}\times{}_0\vec{p})\cdot(\vec{\Omega}\times{}_0\vec{p})\rho\,\mathrm{d}S\right] \\ &+\left[h\dot{\vec{P}}\cdot\int\int_S \dot{\vec{w}}\rho\,\mathrm{d}S + h\vec{\Omega}.\int\int {}_0\vec{p}\times\dot{\vec{w}}\rho\,\mathrm{d}S + \frac{h}{2}\int\int_S \dot{\vec{w}}\cdot\dot{\vec{w}}\rho\,\mathrm{d}S\right] \\ &+\frac{h^3}{24}\left[\int\int_S(\vec{\Omega}\times\hat{c}_3)\cdot(\vec{\Omega}\times\hat{c}_3)\rho\,\mathrm{d}S\right. \\ &\left.+2\int\int_S(\vec{\Omega}\times\hat{c}_3)\cdot(\dot{\vec{\phi}}\times\hat{c}_3)\rho\,\mathrm{d}S + \int\int(\dot{\vec{\phi}}\times\hat{c}_3)\cdot(\dot{\vec{\phi}}\times\hat{c}_3)\rho\,\mathrm{d}S\right]. \end{aligned} \tag{76}$$

The terms of (76) are grouped into three bracketed groups. The first group is indispensable; its terms depend only on the rigid-body motion of the element. The last group represents so-called rotary inertia; these terms are usually negligible in thin shells. The middle group depends on the small relative displacement $\vec{w}$ which is determined by a priori approximation of deformation within the element. Notice, too, that the first group of (76) must predominate as the size of the element diminishes. We conjecture that the first bracketed term of (76) is enough for the analyses of most thin shells.

APPENDIX TO PART I

The decomposition of motion into finite rotation and strain

A particle P has initial and terminal positions defined by the vectors $\vec{r}$ and $\vec{R}$. Particles of the medium in a neighborhood of P are located by curvilinear coordinates θ^i. The same, but convected, coordinates are employed for the displaced particles. The tangent base vectors for initial and terminal lines are denoted by $\vec{g}_i$ and $\vec{G}_i$, respectively; the reciprocal base vectors are denoted by $\vec{g}^i$ and $\vec{G}^i$, respectively.

We require a transformation of $\vec{g}_i$ into $\vec{G}_i$ via the successive steps:

(1) a rigid motion which transports $\vec{g}_i$ to a similar triad $\vec{g}'_i$ followed by

(2) a deformation which carries $\vec{g}'_i$ to the terminal triad $\vec{G}'_i$.

The required decomposition was described by Toupin†. The treatise by Truesdell and Toupin [12] gives a full account of the "fundamental theorem" and its origins. The presentation in these works is completely general, employing both Eulerian and Lagrangian viewpoints and using "two independently selected general curvilinear co-ordinate systems" [12]. Here we are concerned with the motion of a solid and, specifically, an elastic solid. Consequently, we are compelled to use initial coordinates and adopt the Lagrangian viewpoint. Since the Eulerian viewpoint is inappropriate, we can dispense with the dual system of coordinates and use the appropriate simplification which we present below.

Strain tensor

Let $\mathrm{d}s$ and $\mathrm{d}S$ denote the differential lengths of a material line before and after deformation, let g_{ij} and G_{ij} denote covariant components of the metric tensors of the undeformed and deformed systems, and let γ_{ij} denote the strain tensor. The lengths, metric tensors and strain tensor are related as follows:

$$\mathrm{d}S^2 - \mathrm{d}s^2 = (G_{ij} - g_{ij})\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j \equiv 2\gamma_{ij}\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j. \tag{77a, b}$$

Let the triads $\hat{n}_\alpha = n_\alpha \vec{g}_i$ and $\hat{N}_\alpha = N_\alpha \vec{G}_i$ ($\alpha = 1, 2, 3$) define the principal directions of strain in the undeformed and deformed body, respectively, and let e_α and E_α denote the principal values with respect to the initial and terminal systems, respectively. The components of the unit vectors are related as follows:

$$n^i_\alpha = (1+2e_\alpha)^{\frac{1}{2}} N^i_\alpha, \qquad N^i_\alpha = (1-2E_\alpha)^{\frac{1}{2}} n^i_\alpha.$$

A strain component is expressed in terms of the principal values by the following:

$$\begin{aligned}\gamma_{ij} &= \sum_\alpha e_\alpha n_{\alpha i} n_{\alpha j}\\ &= \sum_\alpha E_\alpha N_{\alpha i} N_{\alpha j}.\end{aligned}$$

Rotation tensor

The motion which carries the orthonormal triad $\hat{n}_\alpha$ to the orthonormal triad $\hat{N}_\alpha$ is a rigid motion characterized by an orthogonal transformation which expresses the components of $\hat{N}_\alpha$ in terms of the components of $\hat{n}_\alpha$.

Let $R^{i}_{\cdot j}$ be a two-system tensor which defines the rigid rotation as follows:

$$N^i_\alpha = R^{i}_{\cdot j} n^j_\alpha. \tag{78a}$$

It is a two-system tensor because the first suffix signifies its tensor character in the deformed system (G_{ij}) while the second signifies its tensor character in the initial system (g_{ij}). The inverse of (78a) is

$$n^i{}_\alpha = R^{\cdot i}_{j} N^j_\alpha. \tag{78b}$$

In this instance, it is more meaningful to stay in the initial reference system. Accordingly, the components of $\hat{N}_\alpha$ are expressed in terms of the initial base vectors $\vec{g}_i$:

$$\hat{N}_\alpha = (r^{i}_{\cdot j} n^j_\alpha)\vec{g}_i \tag{79}$$

† R. A. Toupin, *op. cit.*

where $r^{i}_{\cdot j}$ is a tensor with respect to the initial system and related to $R^{i}_{\cdot j}$ as follows:

$$r^{j}_{\cdot i} = (\vec{g}^{j} \cdot \vec{G}_{p})R^{p}_{\cdot i} \tag{80a}$$

$$R^{j}_{\cdot i} = (\vec{G}^{j} \cdot \vec{g}_{p})r^{p}_{\cdot i}. \tag{80b}$$

Motion of the base triad

The base triad $\vec{G}_i$ of the deformed coordinates is given by

$$\vec{G}_i = (\sum_{\alpha} (1+2e_{\alpha})^{\frac{1}{2}} n_{\alpha i} n^{j}_{\alpha})(r^{p}_{\cdot j}\vec{g}_{p}). \tag{81}$$

Clearly, the second parenthetical factor represents a rigid rotation of the base triad $\vec{g}_p$ to

$$\vec{g}'_j \equiv r^{p}_{\cdot j}\vec{g}_{p} \tag{82}$$

while the first parenthetical expression represents a deformation which carries the triad $\vec{g}'_j$ to $\vec{G}_i$. Let us define

$$_{\frac{1}{2}}G^{j}_{i} \equiv \sum_{\alpha} (1+2e_{\alpha})^{\frac{1}{2}} n_{\alpha i} n^{j}_{\alpha}. \tag{83}$$

Note that this tensor has the principal directions of γ_{ij}, but different principal values.

Approximation of small strain

If the principal extensions are small compared to unity, i.e. $e_{\alpha} \ll 1$, then

$$_{\frac{1}{2}}G_{ij} \doteq \sum_{\alpha} (1+e_{\alpha}) n_{\alpha i} n_{\alpha j} = g_{ij} + \gamma_{ij}. \tag{84}$$

A corresponding approximation of (83) leads to the following approximations

$$r^{p}_{\cdot j} \doteq \vec{g}^{p} \cdot \vec{G}_{i}(\delta^{i}_{j} - g^{ik}\gamma_{kj}) \tag{85}$$

$$\vec{G}_i \doteq (\delta^{j}_{i} + \gamma^{j}_{i})\vec{g}_j{}'. \tag{86}$$

Approximation of small strain and moderate rotation

The approximation (86) is limited to small strains, but applies to unrestricted rotations. In most structural problems, the rotations are not unrestricted but small or, at most, moderately large. We regard a rotation as "moderate" if it is small enough to be treated as a vector with adequate accuracy; a rotation with the order-of-magnitude $\frac{1}{10}$ is moderate.

By the foregoing definition of "moderate," the rotated base vector $\vec{g}'_i$ is given by

$$\vec{g}'_i = \vec{g}_i + \vec{\theta} \times \vec{g}_i. \tag{87}$$

It follows from (82) and (87) that the rotation tensor has the form:

$$r_{pj} \doteq g_{pj} + \theta^{k}\varepsilon_{kjp} \equiv g_{pj} + \theta_{pj}. \tag{88a, b}$$

The strain components are now small; the order-of-magnitude of a physical component is $\frac{1}{100}$ or less. The rotation components are moderate; the order-of-magnitude of a physical component of the rotation vector is $\frac{1}{10}$ or less.

In accordance with (88), equation (86) assumes the form

$$\vec{G}_i = \vec{g}_i + \gamma^{j}_{i}\vec{g}_j + \theta^{j}_{\cdot i}\vec{g}_j + \gamma^{m}_{i}\theta^{j}_{\cdot m}\vec{g}_j. \tag{89}$$

In view of the symmetry of g_{ij} and γ_{ij}, and the skewsymmetry of θ_{ij}:

$$e_{ij} \equiv \tfrac{1}{2}(\vec{g}_i \cdot \vec{G}_j + \vec{g}_j \cdot \vec{G}_i) - g_{ij} = \gamma_{ij} + \tfrac{1}{2}(\gamma_j^m \theta_{im} + \gamma_i^m \theta_{jm}) \tag{90}$$

$$\Omega_{ij} \equiv \tfrac{1}{2}(\vec{g}_i \cdot \vec{G}_j - \vec{g}_j \cdot \vec{G}_i) = \theta_{ij} + \tfrac{1}{2}(\gamma_j^m \theta_{im} - \gamma_i^m \theta_{jm}). \tag{91}$$

From (90) and (91) we conclude that first approximations are

$$\gamma_{ij} \doteq e_{ij} \equiv \tfrac{1}{2}(\vec{g}_i \cdot \vec{G}_j + \vec{g}_j \cdot \vec{G}_i) - g_{ij} \tag{92}$$

$$\theta_{ij} \doteq \Omega_{ij} \equiv \tfrac{1}{2}(\vec{g}_i \cdot \vec{G}_j - \vec{g}_j \cdot \vec{G}_i). \tag{93}$$

The approximation (92) neglects terms of order θ compared with unity, while (93) neglects terms of order γ compared with unity. The latter is adequate, while the former is a poor approximation for moderate rotations. To obtain a better approximation, we return to the definition (77b), and obtain

$$\gamma_{ij} = e_{ij} + \tfrac{1}{2}g^{pq}(e_{pi} + \Omega_{pi})(e_{qj} + \Omega_{qj}). \tag{94}$$

We recall that Ω_{ij} of (93) provides a good approximation for moderate rotations while e_{ij} has the order-of-magnitude of a strain component. Accordingly, we neglect the products in (94) containing e_{ij}. The resulting approximation for moderate rotations is

$$\gamma_{ij} = e_{ij} + \tfrac{1}{2}g^{pq}\Omega_{pi}\Omega_{qj}. \tag{95}$$

Approximation of small strain and small rotation

If the strains and rotations are small, say $\frac{1}{100}$ or less, then the first approximations (92) and (93) hold. It follows that

$$\vec{G}_i = \vec{g}_i + (\gamma_{ij} + \theta_{ji})\vec{g}^j \tag{96}$$

In accordance with (87) and (96),

$$\vec{G}_i = \vec{g}_i + \vec{\theta} \times \vec{g}_i + \gamma_i^j \vec{g}_j. \tag{97}$$

Here, the vector $\vec{G}_i$ is the sum of three terms: the original tangent vector ($\vec{g}_i$), the contribution from rigid rotation ($\vec{\theta} \times \vec{g}_i$) and the contribution from deformation ($\gamma_i^j \vec{g}_j$).

Relation between finite rotation and small relative rotation

At each point of the deformed medium we conceive a triad $\vec{g}_i'$ obtained by rigidly transporting the triad $\vec{g}_i$ according to (82). In the event of small strain, the tangent vector of a deformed coordinate line is given by (86).

We may take an alternative viewpoint and conceive a neighborhood V of a point P rigidly transported to V'' with the particle at P moving to its terminal position at P^* and the triad $\vec{g}_i$ at P rigidly transported to $\vec{g}_i'$ at P^*. Moreover, the differential geometry of the neighborhood is unchanged by the rigid motion; at each point of V'' we conceive a triad of tangent vectors $\vec{g}_i''$ at each particle of the displaced neighborhood, similar to the initial triad $\vec{g}_i$ at the same particle, but rotated such that

$$\vec{g}_i''(P^*) = \vec{g}_i'(P^*). \tag{98}$$

The motion which deforms the neighborhood V'' is characterized by small *relative* rotations. The tangent vector $\vec{G}_i$ is expressed in terms of the rotated vectors $\vec{g}_i''$ in the manner of (97), but $\vec{\theta}$ now denotes only the additional rotation of $\vec{g}_i''$ which is caused by the deformation:

$$\vec{G}_i = (\delta_i^j + \gamma_i^j)\vec{g}_j'' + \vec{\theta} \times \vec{g}_i'' \tag{99a}$$

$$\equiv (\delta_i^j + \gamma_i^j + \vec{\theta}^j_{.i})\vec{g}_j''. \tag{99b}$$

Note that the latter viewpoint incorporates the approximations of small strain and small *relative* rotations. This means that the accuracy diminishes as the neighborhood increases.

Inverting (86), we obtain the approximation:

$$\vec{g}_j' = (\delta_j^i - \gamma_j^i)\vec{G}_i. \tag{100}$$

Substituting (99) into (100) and, again, neglecting products of the strain and relative rotation components, we have

$$\vec{g}_j' = \vec{g}_j'' + \vec{\theta} \times \vec{g}_j''. \tag{101}$$

In accordance with (82) and (101)

$$r^p_{.j} = \vec{g}^p \cdot (\vec{g}_j'' + \vec{\theta} \times \vec{g}_j''). \tag{102}$$

Equation (102) has special significance in the analyses of large deflections by means of small finite elements. Suppose, for example, that the rigid rotation is known at a point P and the small relative displacements are known in a small region about P. Then, $\vec{g}_j''$ is determined by the former, $\vec{\theta}$ by the latter and the rotation tensor $r^p_{.j}$ is computed for a neighboring point by (102).

Shifting

Equations (101) and (102) apply at any point Q in a small enough neighborhood of P. Moreover, the base vector $\vec{g}_i''$ is evaluated at the point Q, *but* it is obtained by rotating the base triad $\vec{g}_i$ of Q by the amount of the rotation at P. However, the rotation tensor $r^i_{.j}$ of P is referred to the triad at P. That is,

$$\vec{g}_i''(P) = r^j_{.i}(P)\vec{g}_j(P). \tag{103}$$

The base vector $\vec{g}_i''(Q)$ at Q is expressed in terms of $\vec{g}_i''(P)$ as follows:

$$\vec{g}_i''(Q) = [\vec{g}''^n(P) \cdot \vec{g}_i''(Q)]\vec{g}_n''(P). \tag{104}$$

Substituting (103) into (104), we have

$$\vec{g}_i''(Q) = [\vec{g}''^n(P) \cdot \vec{g}_i''(Q)]r^j_{.n}(P)\vec{g}_j(P). \tag{105}$$

In the manner of (104),

$$\vec{g}_i(P) = [\vec{g}^s(Q) \cdot \vec{g}_i(P)]\vec{g}_s(Q). \tag{106}$$

Substituting (106) into (105), we obtain

$$\vec{g}_i''(Q) = [\vec{g}''^n(P) \cdot \vec{g}_i''(Q)][\vec{g}^s(Q) \cdot \vec{g}_j(P)]r^j_{.n}(P)\vec{g}_s(Q). \tag{107}$$

Recall, however, that the vector $\vec{g}_i''$ is merely a rotated version of $\vec{g}_i$. Consequently, the primes in the first bracket of (107) can be suppressed; then

$$\vec{g}_i''(Q) = [\vec{g}^n(P) \cdot \vec{g}_i(Q)][\vec{g}^s(Q) \cdot \vec{g}_j(P)]r^j_{.n}(P)\vec{g}_s(Q) \tag{108a}$$

$$\equiv r^s_{.i}(PQ)\vec{g}_s(Q). \tag{108b}$$

The bracketed quantities of (108a) are termed "shifters" and serve to shift the tensor of P to the base system of Q. For brevity, we denote the shifted component by $r^{s}_{;i}(PQ)$ as defined by (108b).

PART II: APPLICATION

Synopsis of Part I

The motion of a finite element of a shell is decomposed into a rigid-body displacement followed by a small deformation; the deformation is characterized by small relative displacements. Small relative rotations are related to differences of finite rotations.

A theory for small strain of a shell-element is based upon the hypothesis that a normal remains straight, but not necessarily normal.

The decomposition serves to adapt linear finite-element approximations to nonlinear problems of large rotation. The generalization of the Kirchhoff theory admits simpler approximations of the displacement within an element.

Approximating the deformation of the element

The displacement of the surface $\mathscr{S}$ is to be approximated piecewise by approximation of each element. The pieces must provide a continuous displacement of $\mathscr{S}$. In addition, the edges of adjacent elements must be contiguous.

The deformation of the element V is defined by the relative displacement of the reference surface S and the relative rotation and extension of the normal. In keeping with the plane-stress assumption of (36), (37), and (40), we impose no approximation of the transverse extensional strain ε. Then, the deformation is fully characterized by the three components of $\bar{w}$ in (22) and the two tangential components of the rotation $\vec{\phi} = (\bar{\omega}-2\vec{\beta})$ in (60). The former define the deformation of the reference surface S and the latter define warping of the edges of V.

To be specific, let the base point P have coordinates $\theta^\alpha = a^\alpha$ and, to simplify matters; let

$$x^\alpha = (\theta^\alpha - a^\alpha)/L^{(\alpha)} \tag{109}$$

where L^α is so chosen that the corners of S are at $(x^1, x^2) = (0,0), (0,1), (1,1), (1,0)$.

The simplest polynomial approximations which preserve continuity are based upon the Lagrangian interpolation between inter-element boundaries:

$$w_\alpha = B_{\alpha\beta}x^\beta + C_\alpha x^1 x^2, \qquad (B_{\alpha\beta} = B_{\beta\alpha}) \tag{110a}$$

$$w_3 = C_3 x^1 x^2 \tag{110b}$$

$$\phi_\alpha \equiv \omega_\alpha - 2\gamma_\alpha \tag{111a}$$

$$= D_\alpha + E_{\alpha\beta}x^\beta + F_\alpha x^1 x^2 \tag{111b}$$

Here the forms of (110) are simplified because the relative displacement $\bar{w}$ and rotation $\bar{\omega}$ vanish at the base point ($x^\alpha = 0$). The surface S^* defined by (110) has six degrees-of-*flexibility*. In addition, the rigid-body motion provides six more degrees-of-freedom. Together, the twelve parameters are determined by the displacements of the four corners of the element. Likewise, the eight parameters of (111) are determined by the rotations at the four corners.

The linearity along the edges and coincidence at corners insures the continuity of the surface $\mathscr{S}^*$ and the contiguity of interelement edges. In principle, the approximations (110) and (111) are comparable to Melosh's "flat triangular" approximation [13].

If the Kirchhoff hypothesis is invoked ($\gamma_\alpha = 0$ and $\phi_\alpha = \omega_\alpha$), then the foregoing approximation (110, 111) is unacceptable because adjacent edges are discontiguous. The requisite contiguity is achieved only if the surface has a continuous normal. A Hermitian interpolation [6] of the transverse displacement w_3 is required in S. Then, the component w_3 is approximated by a sixth-degree polynomial and the quadrilateral element has at least 24 degrees-of-freedom.

The approximation (110) describes extensional deformations as well as most, and, together with (111), admits transverse-shear deformations. In addition, the approximation (110b) is far simpler than the Hermitian interpolation. On the other hand, the approximation (110b) has an inherent disadvantage: It provides no curvature-change within the finite element. This means that two elements are needed for the crudest approximation of flexure. Consequently, more elements may be needed wherever flexure predominates as, for example, in a plate, near the edge of a shell or near a region of concentrated loads. Accordingly, we present an alternative which provides changes-of-curvature yet retains some of the simplicity achieved by relaxing the Kirchhoff hypothesis:

As an alternative to (110b) we may take

$$\begin{aligned} w_3 = {} & C_3 x^1 x^2 + \tfrac{1}{2} D_{31}(x^1)^2 + \tfrac{1}{2} D_{32}(x^2)^2 \\ & + \tfrac{1}{2} E_{31}(x^1)^2 x^2 + \tfrac{1}{2} E_{32}(x^2)^2 x^1 \end{aligned} \tag{112}$$

The function (112) is quadratic along an inter-element boundary. The quadratic is determined by the values at the end points and another value, say, the value at the mid-point. The four additional parameters of (112) provide just the required flexibility to maintain continuity at interelement boundaries. The approximation (112) for w_3 and (110a) for w_α, together with (111) provides 18 degrees of flexibility just as the approximation [6] which provides a smooth surface $\mathscr{S}^*$. However, the present approximation utilizes only cubic terms and, therefore, provides the simpler basis for deriving stiffness matrices. Moreover, our approximation admits transverse shear deformation which can be significant, particularly, when the shell is nonhomogeneous as a sandwich.

It is interesting to note that a couple and force on *each* edge of the quadrilateral element form a collection of 24 components on the four edges. This is the number of generalized forces obtained from the approximation of (110a), (111), and (112) or the Hermitian approximation [6] with the Kirchhoff theory.

Kirchhoff-type constraints

According to (111) a material normal rotates independently of the motion of the reference surface S. In particular, each corner of a finite element can rotate independently of the other corners. This freedom is more than needed to insure the contiguity of the edges of adjacent elements. It is the freedom of transverse shear and it can be eliminated by a constraint analogous to the Kirchhoff hypothesis: For example, we may require that the transverse shear γ_α vanish at the mid-point of the θ^α edge. By imposing the condition at the mid-point of each side, we reduce the degrees-of-flexibility by four. In the limit, as the element becomes infinitesimal, our approximation must then approach the Kirchhoff theory. The four constraints constitute the discrete counterpart of the Kirchhoff hypothesis.

Generalized coordinates—local and global

The motion of the finite element is decomposed into a rigid-body motion and a relative motion. The former is defined by the velocity $\dot{\vec{P}}$ of the base particle P and the angular velocity $\vec{\Omega}$ associated with point P. The rigid motion is determined by the three components of $\vec{P}$ and three variables (e.g. angles of rotation) which determine $\vec{\Omega}$; these six time-dependent quantities are here called the *generalized global coordinates.*

The relative motion of any particle of the element is defined by the relative velocity $\dot{\vec{w}}$, the relative angular velocity $\dot{\vec{\phi}}$ and the extensional rate $\dot{\varepsilon}$. The last is to be ignored in favor of the plane-stress assumption. The relative velocities $\dot{\vec{w}}$ and $\dot{\vec{\phi}}$ are determined by the approximation for $\vec{w}$ and $\vec{\phi}$ which serve to express these functions in terms of discrete time-dependent quantities. These quantities are here called the *generalized deformation coordinates* and their time derivatives are called the *generalized deformation* rates. The number of such deformation coordinates determines the degree-of-flexibility.

Let q_N denote a generalized coordinate. If there are 'n' generalized deformation coordinates, then $N = 1, --, n$ signifies a deformation coordinate. The coefficients in (110) and (111) are examples of possible deformation coordinates. The global coordinates are signified by $N = n+1, --, n+6$.

Any infinitesimal movement of the system is defined by '$n+6$' virtual 'displacements' δq_N. The work of external forces upon the system takes the form:

$$\delta w = \sum_{N=1}^{n+6} p_N \delta q_N \tag{113}$$

The variable p_N is the generalized force associated with the coordinate q_N. The generalized forces are comprised of forces t_N exerted by external agencies and forces f_N exerted by adjacent elements via contiguous edges. Such generalized forces need not be physical forces or couples.

To achieve continuity of the displacements and reactions along inter-element boundaries, we express the incremental displacements δq_N and forces f_N in terms of identifiable displacements and actions at the edges: Our choice of displacements depends upon the approximations of w_i and ϕ_α. If a polynomial approximation of the transverse displacement w_3 contains 'k' parameters, we require $k+3$ edge values; three additional values account for the rigid-body motion. Likewise, if polynomial approximations of w_α contain 'l' parameters, then we require '$l+3$' edge values. If we restrict our attention to the approximation (111) for ϕ_α, then the 8 components of the corner-rotations are suitable alternatives to the coefficients in (111). Note that the total of generalized coordinates is

$$n+6 = (k+3)+(l+3)+8$$

and the number of deformation coordinates is

$$n = k+l+8$$

The equations which express the generalized coordinates in terms of identifiable edge displacements are obtained as follows:

Let $x^\alpha = p^\alpha$ locate a particle ${}_0Q^*$ at the edge of S^* and let ${}_0\vec{p}$ be the vector $\overline{P^*{}_0Q^*}$. Then a virtual displacement of the particle ${}_0Q^*$ is

$$\overrightarrow{\delta V}(p^1, p^2) = \overrightarrow{\delta V}(0,0) + \overrightarrow{\delta\alpha} \mathbf{x}\, {}_0\vec{p} + \overrightarrow{\delta w}(p^1, p^2)$$

where $\overrightarrow{\delta\alpha}$ is the rigid-body rotation of the element. At each of '$k+3$' points along the edge of S^*,

$$\delta V_{3K} \equiv \hat{c}_3 \,.\, \overrightarrow{\delta V_K} = \hat{c}_3 \,.\, [\overrightarrow{\delta V}(0,0) + \overrightarrow{\delta\alpha} \mathbf{x}_0 \vec{p} + \overrightarrow{\delta w_K}] \qquad K = 1, \ldots, k+3 \tag{114a}$$

wherein the subscript 'K' signifies a certain point of the edge. At each of '$(l+3)/2$' points along the edge of S^*,

$$\delta V_{\alpha L} \equiv \vec{c}_\alpha \,.\, \overrightarrow{\delta V_L} = \vec{c}_\alpha \,.\, [\overrightarrow{\delta V}(0,0) + \overrightarrow{\delta\alpha} \mathbf{x}_0 \vec{p} + \overrightarrow{\delta w_L}] \qquad L = 1, \ldots, (l+3)/2 \tag{114b}$$

In addition to (114a, b), there are 8 equations which assert that the rotation of a corner is the sum of the rigid-body rotation $\overrightarrow{\delta\alpha}$ and the relative rotation $\overrightarrow{\delta\phi}$:

$$\delta\psi_{\alpha M} \equiv \vec{c}_\alpha \,.\, \overrightarrow{\delta\psi_M} = \vec{c}_\alpha \,.\, [\overrightarrow{\delta\alpha} + \overrightarrow{\delta\phi_M}] \qquad M = 1, 2, 3, 4 \tag{114c}$$

The total number of (114a, b, c) is '$n+6$', the number of generalized coordinates. The left sides of (114a, b) and (114c) are components of displacement at edge-points of S^* and of rotations at the corners of V^*. The right sides are linear combinations of the generalized displacements δq_N, i.e.

$$\delta V_{3K} = \sum_{N=1}^{n+6} A_{KN} \delta q_N \tag{114d}$$

$$\delta V_{\alpha L} = \sum_{N=1}^{n+6} B_{\alpha LN}\, \delta q_N \tag{114e}$$

$$\delta\psi_{\alpha M} = \sum_{N=1}^{n+6} C_{\alpha MN}\, \delta q_N \tag{114f}$$

Note: Not all generalized coordinates appear on the right sides of (114); those defining relative edge rotations do not enter δV_{3K} nor $\delta V_{\alpha L}$ and those defining relative displacements do not enter $\delta\psi_{\alpha M}$.

The inverse of (114) expresses the generalized displacements as linear combinations of the edge displacements and rotations, viz.

$$\delta q_N = \sum_{K=1}^{k+3} \tilde{A}_{NK}\, \delta V_{3K} + \sum_{L=1}^{(l+3)/2} \tilde{B}^\alpha_{NL}\, \delta V_{\alpha L} + \sum_{M=1}^{4} \tilde{C}^\alpha_{NM}\, \delta\psi_{\alpha M}, \qquad N = 1, \ldots, n+6 \tag{115}$$

If the rigid rotation of the element is moderate, then (114) and (115) serve as well to relate coordinates as virtual displacements.

In the event of small rotations, edge displacements V_{3K}, $V_{\alpha L}$ and rotations $\psi_{\alpha M}$ may prove to be the most convenient generalized coordinates. Then, we need only express the approximations w_i, ϕ_α in terms of the edge displacements and rotations and identify each of the latter with a label $q_N (N = 1, --, n+6)$.

Generalized edge reactions

Let f^3_K denote force components acting at the '$k+3$' edge locations of virtual displacement δV_{3K}, let f^α_K denote the force components at the '$(l+3)/2$' locations of virtual displacement $\delta V_{\alpha L}$ and let m^α_M denote couple components at the corners of the element. The virtual work of these forces is

$$\delta w = \sum_{K=1}^{k+3} f_K^3 \, \delta V_{3K} + \sum_{L=1}^{(l+3)/2} f_L^\alpha \, \delta V_{\alpha L} + \sum_{M=1}^{4} m_M^\alpha \, \delta\psi_{\alpha M} \tag{116a}$$

$$= \sum_{N=1}^{n+6} f_N \, \delta q_N \tag{116b}$$

Equations (114) may be used to express δV_{3K}, $\delta V_{\alpha L}$ and $\delta\psi_{\alpha M}$ on the left side in terms of δq_N, or (115) may be used to express the δq_N on the right in terms of δV_{3K}, $\delta V_{\alpha L}$ and $\delta\psi_{\alpha M}$. In the first case, since the δq_N are independent, equation (116) provides '$n+6$' equations:

$$f_N = \sum_{K=1}^{k+3} A_{KN} f_K^3 + \sum_{L=1}^{(l+3)/2} B_{\alpha LN} f_L^\alpha + \sum_{M=1}^{4} C_{\alpha MN} m_M^\alpha \tag{117}$$

If (116b) is expressed in terms of δV_{3K}, $\delta V_{\alpha L}$ and $\delta\psi_{\alpha M}$ by means of (115), then, since these displacements are independent, equation (116) provides '$n+6$' equations:

$$f_K^3 = \sum_{N=1}^{n+6} \tilde{A}_{NK} f_N \tag{118a}$$

$$f_L^\alpha = \sum_{N=1}^{n+6} \tilde{B}_{NL}^\alpha f_N \tag{118b}$$

$$m_M^\alpha = \sum_{N=1}^{n+6} \tilde{C}_{NM}^\alpha f_N \tag{118c}$$

Of course, (118) is the inverse of (117).

Generalized external loads

Let point Q^* mark a particle on the upper (or lower) surface of V^*, let $\vec{p}$ denote the vector $\overline{P^*Q^*}$, let $\vec{t}$ denote the traction exerted on the surface and let $\overline{\delta V}$ denote a virtual displacement at Q^*. Then the virtual work of the applied traction is

$$\delta w \equiv \vec{t} \, . \, \overrightarrow{\delta V}$$

$$= \vec{t} \, . \left[\delta V(0,0) + \overrightarrow{\delta\psi} \times \vec{p} + \overrightarrow{\delta w} + \overrightarrow{\delta\phi} \times \hat{c}_3 \frac{h}{2} \right]$$

For simplicity, we suppose that the traction is referred to the initial area S. Then the virtual work of all tractions upon the surface is

$$\begin{aligned} \delta w = \overrightarrow{\delta V}(0,0) \cdot \int\int_S \vec{t} \, \mathrm{d}S + \int\int_S \vec{t} \cdot \overrightarrow{\delta w} \, \mathrm{d}S \\ + \overrightarrow{\delta\psi} \cdot \int\int_S \vec{p} \times \vec{t} \, \mathrm{d}S + \frac{h}{2} \hat{c}_3 \cdot \int\int_S \vec{t} \times \overrightarrow{\delta\phi} \, \mathrm{d}S \end{aligned} \tag{119a}$$

With the selection of global coordinates, the rigid-body displacements $\overrightarrow{\delta V}(0,0)$ and $\overrightarrow{\delta\psi}$ can be expressed in terms of generalized displacements $\delta q_N (N = n+1, --, n+6)$. The relative displacements $\overrightarrow{\delta w}$ and $\overrightarrow{\delta\phi}$ are expressed in terms of generalized displacements

$\delta q_N (N = 1, --, n)$ by the approximations. Then equation (119a) assumes the form

$$\delta w = \sum_{N=1}^{n+6} t_N \delta q_N \tag{119b}$$

wherein the coefficients t_N are certain weighted integrals of t^i. We call the coefficients t_N the generalized external loads.

Again, certain edge displacements and rotations can serve as generalized coordinates, particularly when rotations are small. Then the generalized external loads can be identified with actions at prescribed points of the edge.

Approximate equations of motion for the finite element

With our choice of global coordinates to define rigid-body motion and with our approximation of the functions w_i and ϕ_α the velocities $\dot{\vec{P}}$, $\vec{\Omega}$, $\dot{\vec{w}}$ and $\dot{\vec{\phi}}$ can be expressed in terms of the generalized velocities $\dot{q}_N$ and coordinates q_N. Then, by means of (76) the kinetic energy can be expressed in terms of the generalized variables:

$$T = T(\dot{q}_N, q_N) \tag{120}$$

Only three coordinates q_N, the global coordinates defining the rigid-body rotation, appear in the kinetic energy in addition to the derivatives $\dot{q}_N$.

Equations (25) and (28) express the surface strain ${}_{\Omega}\gamma_{\alpha\beta}$ in terms of the relative displacement $\vec{w}$. Equations (24) and (60) serve to express the transverse shear strain γ_α in terms of the relative displacement $\vec{w}$ and rotation $\vec{\phi}$. Then, with the aid of (24), (26), (35b) and (20), the variable $\eta_{\alpha\beta}$ can be expressed in terms of the variable $\vec{w}$ and $\vec{\phi}$. Summarizing these kinematical relations, we have

$${}_{\Omega}\gamma_{\alpha\beta} = \tfrac{1}{2}(w_\alpha|_\beta + w_\beta|_\alpha - 2w^3 b_{\alpha\beta}) \tag{121a}$$

$$\gamma_\alpha = \tfrac{1}{2}(\varepsilon_{\alpha\beta}\phi^\beta + w_{3,\alpha} + w^\lambda b_{\lambda\alpha}) \tag{121b}$$

$$\eta_{\alpha\beta} = \gamma_\alpha|_\beta + \gamma_\beta|_\alpha - \varkappa_{\alpha\beta} \tag{121c}$$

$$\begin{aligned} \varkappa_{\alpha\beta} = {} & \tfrac{1}{2}[w_3|_{\alpha\beta} + w_3|_{\beta\alpha} + (w^\lambda b_{\lambda\alpha})|_\beta + (w^\lambda b_{\lambda\beta})|_\alpha \\ & + \tfrac{1}{2}(w_\alpha|_\mu - w_\mu|_\alpha) b^\mu_\beta - \tfrac{1}{2}(w_\beta|_\mu - w_\mu|_\beta) b^\mu_\alpha] \end{aligned} \tag{121d}$$

It follows that the strain-energy density (39) can be expressed in terms of the variables $\vec{w}$ and $\vec{\phi}$ and, after the approximation of the functions $\vec{w}$ and $\vec{\phi}$, the strain-energy integral (40) can be expressed in terms of 'n' *generalized deformation coordinates*:

$$U = U(q_N)$$

The generalized loads t_N are obtained via (119).

Finally, the equations of motion for the finite-element are the Lagrange equations [14]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_N}\right) - \frac{\partial T}{\partial q_N} + \frac{\partial U}{\partial q_N} - t_N - f_N = 0 \tag{122}$$

$$N = 1, \ldots, n, \ldots, n+6$$

Equations (122) relate the generalized coordinates q_N and edge reactions f_N. Since the coordinates $q_{n+1}---q_{n+6}$ define the rigid-body motion, we anticipate that the corresponding equations ($N = n+1,--,n+6$) govern the gross motion. Conversely, since the coordinates q_1--q_n define the deformation, we anticipate that the corresponding equations ($N = 1,--,n$) are constitutive relations.

In the intrinsic theory of shells the governing equations reveal no measure of any finite rotation. This happy circumstance occurs because the equations are referred to the current directions of the convecting lines. Analogous results are achieved in equations (122) if the Lagrange equations are formulated with global coordinates ($N = n+1,--,n+6$) and velocities referred to the convected triad $\hat{c}_i$. In particular, the global orientation can be defined by rotations about the directions of the base vectors $\vec{c}_i$.

For each element of the shell there are '$n+6$' equations (122) which contain '$2(n+6)$' variables $q_N(t)$ and $f_N(t)$. The additional equations are obtained by equating the displacements and/or rotations of certain coincident particles or lines of continuous edges and by requiring that the sum of all actions and reactions on such particles and/or lines vanish:

Geometrical continuity at inter-element boundaries

Let us formulate the condition for equality of displacement at a particle on an inter-element line of the reference surface $\mathscr{S}^*$. In Fig. 5, adjacent elements A and B contain the surfaces S_A and S_B. We imagine that the surfaces are rigidly transported to S''_A and S''_B and then deformed to the contiguous surfaces S^*_A and S^*_B. We require that the displacement at O on S_A be equal to the displacement at O of S_B.

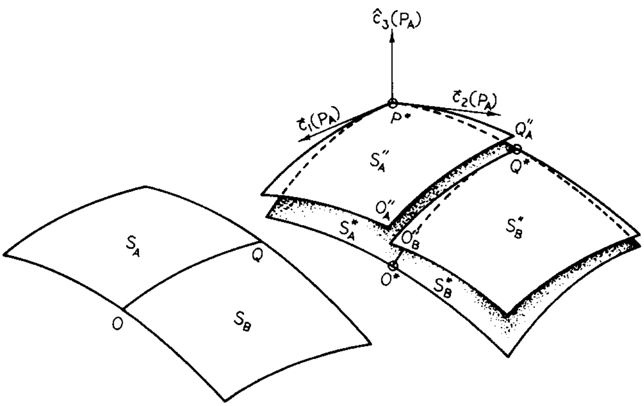

FIG. 5

The rigid motion which carries S_A to S''_A and S_B to S''_B is not relevant, only the difference in the two rigid motions. The difference is characterized by a *small* displacement and a *small* rotation. If P^* and Q^* are the base points of S^*_A and S^*_B, respectively, then the rigid motions differ by the amount of the small displacement $\vec{w}(Q_A)$ and rotation $\vec{\omega}(Q_A)$.

The condition for continuity at O^* is

$$\vec{w}(O_A)-\vec{w}(O_B)-\vec{w}(Q_A)-\vec{\omega}(Q_A)\times\overrightarrow{Q''_AO''_A} = 0 \tag{123}$$

Assuming small strains and sufficiently small elements, we set

$$\overrightarrow{QO} \equiv Q^i\vec{a}_i(Q), \qquad \overrightarrow{Q''_AO''_A} = Q^i\vec{c}_i(Q_A) \tag{124a, b}$$

To express the condition (123) in terms of the components, we must note that the base vectors of S''_A and S''_B differ by the amount of a small rigid-body rotation $\vec{\omega}(Q_A)$. Thus, the displacement at particle O of element B is

$$\vec{w}(O_B) \equiv w^i(O_B)\vec{c}_i(O_B) \tag{125a}$$

$$= w^i(O_B)[\vec{c}_i(O_A) \underline{+\vec{\omega}(Q_A) \times \vec{c}_i(O_A)}] \tag{125b}$$

By means of (124b) and (125b), the terms of equation (123) can be referred to the base triads $\vec{c}_i(Q_A)$ and $\vec{c}_i(O_A)$ of element A. Then the various terms are evaluated at different points and also referred to the base triads of different points. To treat the components we refer all vectors to a common base system; this requires 'shifting' some components as described in the Appendix of Part I. If $\vec{w}(Q)$ is a vector evaluated at point Q, its components referred to the base triad $\vec{c}_i(O)$, or $\vec{c}^{\,i}(O)$, are

$$w^i(Q \to O) = [\vec{c}_j(Q)\,.\,\vec{c}^{\,i}(O)]w^j(Q)$$

$$w_i(Q \to O) = [\vec{c}^{\,j}(Q)\,.\,\vec{c}_i(O)]w_j(Q)$$

The bracketed factors are the so-called 'shifters':

$$[\vec{c}_j(Q)\,.\,\vec{c}^{\,i}(\mathrm{O})] \equiv C_j^{\cdot i}(QO) = C^i_{\cdot j}(OQ) \tag{126a, b}$$

In accordance with (29), we note that

$$\omega^i = \tfrac{1}{2}\varepsilon^{ijk}\omega_{kj}, \qquad \omega_{ji} = \omega^k\varepsilon_{kij} \tag{127a, b}$$

With the notations of (124), (126) and (127) and with the aid of (125), the scalar product of (123) and $\vec{c}_k(O_A)$ is

$$\begin{aligned} &w_k(Q_A) - w_k(O_B) - C^j_{\cdot k}(QO)w_j(Q_A) \\ &\quad -\tfrac{1}{2}[C^i_{\cdot q}(OQ)Q^q \underline{+w^i(O_B)}]\,[C^j_{\cdot p}(OQ)\omega_{ml}(Q_A)\varepsilon^{plm}(Q)\varepsilon_{jik}(O)] = 0 \end{aligned} \tag{128}$$

It is important to note that the continuity requirement does not involve a measure of any finite rotation, but it does contain the nonlinear terms underlined in (125b) and (128). These terms are quadratic in the relative displacements.

The condition for contiguity of edges requires equality of the rotations of coincident normals at certain points of the edge. Let point O of Fig. 5 be a location at which we require coincidence of the normals to S^*_A and S^*_B. Again, only the relative rotations are relevant. The rotation of the normal to S^*_A at O^* *relative* to that at P^* is $\vec{\phi}(O_A)$. The rotation of the normal to S^*_B at O^* *relative* to that at Q^* is $\vec{\phi}(O_B)$. But, the rotation of the normal at Q^* *relative* to P^* is $\vec{\omega}(Q_A)$. Coincidence of the normals to S^*_A and S^*_B at O^* requires that

$$\vec{\phi}(O_A) - \vec{\omega}(Q_A) - \vec{\phi}(O_B) = 0 \tag{129a}$$

or

$$\begin{aligned} &\phi_\alpha(O_A) - \phi_\alpha(O_B) - \tfrac{1}{2}C_{j\alpha}(QO)\omega_{pq}(Q_A)\varepsilon^{jqp}(Q) \\ &\quad -\tfrac{1}{2}\phi^i(O_B)[C^j_{\cdot p}(OQ)\omega_{ml}(Q_A)\varepsilon^{plm}(Q)\varepsilon_{ji\alpha}(O)] = 0 \end{aligned} \tag{129b}$$

The conditions (128) and (129) must be enforced at prescribed points of the inter-element boundaries. For example, at an interior element deformed according to (110) and (111), equations (128) and (129) are imposed at each corner, thus providing 20 conditions, the number of degrees-of-freedom.

The conditions (128) and (129) are the compatibility equations of the discrete system. They play a role analogous to the compatibility equations of continuous shells. Specifically, (128) is analogous to the Gauss equation [8] of the deformed surface. Whereas the compatibility equation of the continuous shell [15] contains products of the changes-of-curvature and strains, equation (128) contains products of relative rotations and relative displacements. Neither involves a measure of finite rotation.

We note that matching the approximations for w_i along inter-element boundaries does not insure precise continuity because the components of adjacent elements are referred to different bases; the bases are slightly rotated e.g. $\vec{c}_i(O_A)$ vs. $\vec{c}_i(O_B)$. Although the surface is connected at discrete points by (128), it may be discontinuous at intermediate points, but only by the amount of higher-order terms like those underlined in (128). When the dimensions of the element are diminished the error diminishes as the product of relative rotations and displacements. In any event, matching components at the inter-element boundaries achieves continuity in the context of the linear theory which merely requires small relative rotations.

The compatibility conditions for point O, (128) and (129), can be expressed in terms of the '$2n$' generalized deformation coordinates q_N of elements A and B by means of (24), (26) and the approximations for ϕ_α and w_i.

Reactive conditions at inter-element boundaries

The '$n+6$' generalized forces f_N are expressed in terms of '$n+6$' resultants f^i_N and m^α_N which act upon designated particles and normals at prescribed points of the elemental edges. Particles and normals so-designated shall be called edge-points and edge normals. They may be common to two, three or four contiguous elements. If they do not lie on an edge of the shell, we shall call them *inter-points* and *inter-normals*. If they lie on the edge of the shell, we shall call them *boundary-points* and *boundary normals*.

At an inter-point of the shell the sum of the forces exerted by all contiguous elements must vanish and at an inter-normal the sum of the couples must vanish in accordance with Newton's third law.

The components f^i_N of the reactive forces exerted upon the inter-point N are expressed in terms of the generalized edge forces by (118a, b). However, the reactive force upon each element is referred to the base triad $\vec{c}^i$ of that element. Accordingly, the reactive forces upon an inter-point must be referred to a common base. The choice of the reference basis is a matter of convenience. For example, suppose that O of Fig. 5 is an inter-point. The force upon the element A is referred to the triad $\vec{c}_i(O_A)$ in accordance with our small-rotation view toward the behavior of element A. The forces upon element B at O are referred to the triad $\vec{c}_i(O_B)$ which differs from $\vec{c}_i(O_A)$ by the small relative rotation $\vec{\omega}(Q_A)$. Then the sum of the reactive forces imposed by the two elements (A and B) is

$$f^i_{O_A}\vec{c}_i(O_A)+f^i_{O_B}[\vec{c}_i(O_A)+\vec{\omega}(Q_A)\times\vec{c}_i(O_A)] \tag{130a}$$

Again, the components of the vector $\vec{\omega}(Q_A)$ can be shifted to the base at O_A. Then, the components of (130a) are

$$f^i_{O_A}+f^i_{O_B}+\tfrac{1}{2}f_{O_Bk}C_{jp}(OQ)\omega_{ml}(Q_A)\varepsilon^{pml}(Q)\varepsilon^{jki}(O) \tag{130b}$$

If point O is an inter-point and a corner, then it is likely to be common to 4 elements as depicted in Fig. 6. Here points P, Q, R and O are the base points of elements A, B, D and C,

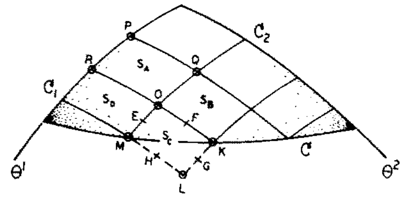

FIG. 6

respectively. The reactive condition at the inter-point 'O' takes the form:

$$\begin{aligned} & f^i_{O_A}+f^i_{O_B}+f^i_{O_C}+f^i_{O_D} \\ &\quad +\tfrac{1}{2}[f_{O_Bk}C_{jp}(OQ)\omega_{ml}(Q_A)\varepsilon^{pml}(Q) \\ &\quad + f_{O_Ck}a_{jp}\omega_{ml}(O_A)\varepsilon^{pml}(O) \\ &\quad +f_{O_D}C_{jp}(OR)\omega(R_A)\varepsilon^{pml}(R)]\varepsilon^{jki}(O) = 0 \end{aligned} \tag{131a}$$

If point 'O' lies on an inter-normal, then we must impose a reactive condition on the couples which act upon the contiguous corners. The condition is similar to (131a) but f^i is replaced by m^α:

$$m^\alpha_{O_A} + \ldots = 0 \tag{131b}$$

Again, we can observe the analogy between the algebraic equations of our discrete system and the equations of the continuous system. Equations (131a) contain products of the forces and *relative* rotations whereas the equations of motion for a shell contain products of the forces and the changes-of-curvature ($\varkappa_{\alpha\beta}n^{\alpha\beta}$, $\varkappa^\alpha_\beta q^\beta$). The relative rotation plays the role of the curvature-change. Likewise, the equations of motion for a continuous shell contain products of couples and changes-of-curvature ($\varkappa_{\alpha\beta}m^{\alpha\beta}$) analogous to the product terms of (131b).

The equations of motion (122), the compatibility conditions (128) and (129), and the reactive conditions (131) describe the motion and connection of interior elements. Only the conditions at boundary-points and boundary-normals are needed to complete the system.

Boundary conditions and boundary elements

The reference surface of a shell is depicted in Fig. 6. Segments $\mathscr{C}_1$ and $\mathscr{C}_2$ of the boundary lie on coordinate lines; segment $\mathscr{C}$ is an arbitrary smooth curve.

Corner 'O' is a typical inter-point. If we employ the approximation (112), then the intermediate points 'E' and 'F' are also inter-points at which we enforce the continuity of transverse displacement.

Point 'R' is typical of the boundary-points which lie on a coordinate curve at the corners of two elements. Continuity of displacement and continuity of rotation of the normal must be enforced by means of (128) and (129), just as it is enforced at an inter-point. In addition, the displacement and rotation may be prescribed. The equations which prescribe displacements and rotations are geometric constraints. Collectively, these constraints comprise the kinematic boundary conditions.

If a Kirchhoff-type constraint is imposed upon the shear deformations (as described previously), then rotation of the normal is not independent of the deflection of the reference surface; rotations in the edge (ϕ_2 at point R) cannot be assigned.

If a geometric constraint is not present at a boundary-point, the corresponding edge force must be prescribed. For example, if there is no constraint on the displacement, then

$$\vec{f}_{R_A} + \vec{f}_{R_D} = 0 \tag{132a}$$

The equations which prescribe forces and couples constitute the dynamic boundary conditions.

A condition which relates displacements and forces (or rotations and couples) may replace a kinematic or dynamic condition. For example, if the boundary point 'R' is attached to a linearly elastic support, then we have

$$\vec{f}_{R_A} + \vec{f}_{R_D} = -k\vec{V}_R \tag{132b}$$

Along the smooth curve $\mathscr{C}$ it is always possible to choose spacings such that the corners of elements at the boundary fall upon the curve. This creates a succession of triangular elements like 'S_C'. The approximation of relative displacements and rotations in the triangular element can be simpler than the approximation in the quadrilateral element, as the former need only maintain continuity along two edges. Indeed, the approximation must be simpler since the number of edge-points and edge-normals is reduced. For example, if the approximations (110a, b) are employed for the quadrilateral elements S_B and S_D, then the corresponding approximations for the triangular boundary-element 'S_C' are

$$w_\alpha = B_{\alpha\beta}x^\beta \qquad (\beta_{\alpha\beta} = B_{\beta\alpha}) \tag{133a}$$

$$w_3 = 0 \tag{133b}$$

The terms suppressed are those which would be needed to match the displacement at the missing corner 'L' and hence, along the edges KL and LM. The simplification of (112) for the triangular boundary-element 'S_C' is

$$w_3 = \tfrac{1}{2}D_{31}(x^1)^2 + \tfrac{1}{2}D_{32}(x^2)^2 \tag{134}$$

The suppressed terms are those which would serve to match the displacement at the missing points G, L and H.

On the presence of finite rotation

We observed previously that the equations of motion (122) need not involve finite rotations or displacements. They are absent when the motion is referred to the current state rather than the initial state, i.e. to the triad $\vec{c}_i$ rather than $\vec{a}_i$. We noted too that the compatibility equations (128) and (129), and the reactive conditions (131) involve only relative rotations. However, loads or boundary conditions may depend on fixed directions rather than the orientation of the element. For example, a gravitational load rather than a fluid pressure, or a fixed edge rather than a free edge. In such cases, it is necessary to introduce the rotation tensor and to call upon the relation (64). Again, this is necessary only if rotations are finite; otherwise there is no need to decompose the rotation into global and relative parts.

Illustrative example I

The simple approximations (110) and (111) and the Kirchhoff-type constraint can be illustrated with a beam. For this purpose, it suffices to consider the linear problem of small deflections caused by a transverse load.

In the one-dimensional version of (110), the relative displacement w_3 vanishes i.e. the middle line of an element undergoes a rigid motion. It follows that the relative rotation $\vec{\omega}$ vanishes also. The rotation $\vec{\phi}$ of the normal relative to the convected triad $\vec{c}_i'$ is then the rotation $-2\vec{\beta}$ caused by shear in accordance with (57), (59), and (60).

Since the indices are not needed in the one-dimensional case, they can be replaced by a suffix 'N' to indicate the element in question. In accordance with (110) and (111), the transverse displacement V_N, shear strain γ_N and rotation ψ_N of the normal within the Nth element are given by the approximations

$$V_N(x) = V_N + \alpha_N x \tag{135}$$

$$\gamma_N(x) = \gamma_N + \beta_N x \tag{136}$$

$$\psi_N(x) = \alpha_N - 2(\gamma_N + \beta_N x) \tag{137}$$

here x is the local coordinate originating at the left end of the Nth element and l is the length of the element.

The cross-sections at the right end of the Nth element and the left of the $(N+1)$th element are contiguous only if

$$\psi_N(l) = \psi_{N+1}(0)$$

It follows from (137) that

$$\alpha_{N+1} - \alpha_N = 2(\gamma_{N+1} - \gamma_N) - 2l\beta_N \tag{138}$$

Likewise, from (135) we have the condition for continuity of displacement:

$$V_{N+1} - V_N = l\alpha_N \tag{139}$$

The constitutive equations for a linearly elastic element with rectangular cross-section follow:

$$\beta_N = \frac{3}{Ebh^3(1+\alpha)}(M_{N+1} + M_N) \tag{140}$$

$$\gamma_N = \frac{1}{2Gblh(1+\alpha)}[(1-2\alpha)M_{N+1} - (1+4\alpha)M_N] \tag{141}$$

wherein, E and G denote the extensional and shear moduli, h the depth, b the width, l the length of each element, M_N the couple at the left end of the Nth element and

$$\alpha = Gl^2/Eh^2 \tag{142}$$

The left side of (140) is the difference between the rotations at the end-sections of the Nth element; the constitutive equation (141) is the counterpart of the moment-curvature relation of the Bernoulli–Euler theory of beams. The constitutive equation (141) determines the transverse-shear deformation.

Now, consider a cantilever beam fixed at the left end and subjected to a load P at the right end. At the left end

$$V_1(0) = 0, \qquad \psi_1(0) = \alpha_1 - 2\gamma_1 = 0 \tag{143a, b}$$

If 'M' denotes the number of elements of length 'l', then

$$M_N = -(M-N+1)Pl \tag{144}$$

In accordance with (144), equations (140) and (141) yield the results:

$$\beta_N = -\frac{6lP}{Ebh^3(1+\alpha)}(M-N+\tfrac{1}{2}) \tag{145}$$

$$\gamma_N = \frac{P}{2Gbh}\left[1+\frac{6\alpha}{1+\alpha}(M-N+\tfrac{1}{2})\right] \tag{146}$$

From (138), (145) and (146) we obtain

$$\alpha_{N+1}-\alpha_N = \frac{12lP}{Ebh^3(1+\alpha)}(M-N) \tag{147}$$

The end shear γ_1 is given by (146) and then the end rotation α_1 by (143b); the rotations $\alpha_N(N = 2, \ldots, M)$ are determined by (147) and then the displacements $V_N(N = 2, \ldots, M+1)$ are given by (139) together with (143a) The numerical values of Table 1 are the dimensionless deflection at the end of the cantilever beam:

$$v = \frac{Eb}{4P}\left(\frac{h}{L}\right)^3 V_3\bigg|_{\theta^2 = L}$$

As the Kirchhoff-type constraint, we take

$$\gamma_N\left(\frac{l}{2}\right) = 0 \tag{148a}$$

According to (136), this implies that

$$2\gamma_N = -l\beta_N \tag{148b}$$

We note that (148) results if γ_N is chosen to minimize the energy due to transverse shear.
With (148b) the strain-energy takes the form

$$U_N = \frac{Elbh^3}{6}(1+\alpha)\beta_N^2 \tag{149}$$

The resulting constitutive equation is again (140). Equations (145) and (147) apply as before. The only difference in the Kirchhoff-type theory is the absence of the constitutive equations (141) or (146) in favor of the constraint (148). In the computations, this difference manifests itself only in the end-condition (143b) wherein γ_1 is now determined by (148b) instead of (146). This circumstance is analogous to the continuum solution: The differential equations of the shear-deformation and Kirchhoff theories are the same. The solutions differ because of the small rotation of the middle line in the amount of the uniform shear.

In the constitutive equations (140), (141), (145), (146), and (147) the factor α tends to zero as $(l/h)^2$. Unless $l/h \ll 1$, the factor has a marked influence on the numerical results. This is evident in columns 'a' and 'b' of Table 1; reasonable results are indicated for values $l/h \leq \frac{1}{2}$. There is a good reason: The factor α stems from the energy of shear deformation.

TABLE 1. END DEFLECTION OF CANTILEVER BEAM UNDER END LOADING

	Finite element theories							Continuum theories			
	a. Shear-deformation			b. Kirchhoff-type			c. Complete Kirchhoff-type	d. Shear-deformation			e. Kirchhoff
L/I \ L/h	6	12	24	6	12	24	All values	6	12	24	All values
2	—	—	—	—	—	—	0·93750	—	—	—	1
4	—	—	—	—	—	—	0·98437	—	—		1
6	0·741	0·402	0·143	0·722	0·397	0·142	0·99306	1·019	1·005	1·001	1
12	0·931	0·731	0·400	0·912	0·726	0·399	0·99826	1·019	1·005	1·001	1
24	0·995	0·919	0·732	0·977	0·914	0·731	0·99957	1·019	1·005	1·001	1
48	1·013	0·982	0·915	0·994	0·977	0·914	0·99989	1·019	1·005	1·001	1
96	—	1·000	0·978	—	0·995	0·977	0·99997	1·019	1·005	1·001	1
192	—	—	0·995	—	—	0·994	0·99999	1·019	1·005	1·001	1

Under the crude approximation (135), which prescribes only rigid rotation of the middle line, considerable shearing must occur. There appears no rational way to suppress the factor α in a shear-deformation theory. However, it is entirely consistent in a Kirchhoff-type theory to neglect the strain-energy caused by shear-deformations. We shall call the result a *complete Kirchhoff-type* theory. Column 'c' of Table 1 shows the excellent results obtained with the latter theory.

Illustrative example II

A key feature of the foregoing theory is the decomposition which enables us to adapt the linear analysis of the finite element to the analysis of geometrically nonlinear problems of flexible bodies. This feature is displayed in a study of the buckling and post-buckling behavior of the column depicted in Fig. 7. Since the linear analysis of the element requires

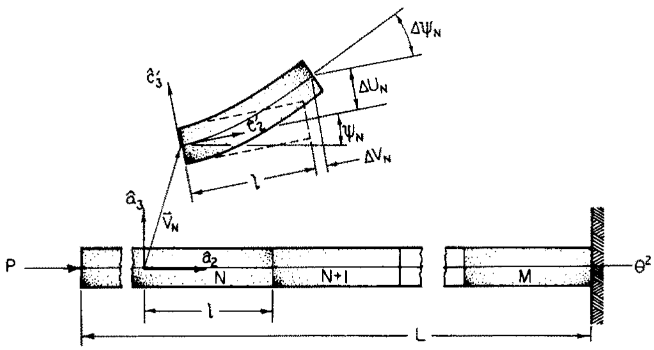

FIG. 7

no approximation, the results indicate only the validity and effectiveness of the decomposition as a tool for handling the geometrical nonlinearities.

For the global coordinates of the Nth element we take the components V_N and U_N of the displacement vector:

$$\vec{V}_N \equiv V_N \hat{c}'_2 + U_N \hat{c}'_3 \tag{150a}$$

$$\equiv X_N \hat{a}_2 + Y_N \hat{a}_3 \tag{150b}$$

and the angle ψ_N shown in Fig. 7.

If we employ the Bernoulli–Euler theory of beams, then there is no need for any approximation of the deformation. The deformation of the element is determined by the relative displacement and rotation of its ends; that is, ΔV_N, ΔU_N and $\Delta\psi_N$ of Fig. 7 serve as the deformation coordinates.

We express the actual force and couple upon the left end of the Nth element as

$$\vec{F}_N \equiv N_N \hat{c}'_2 + Q_N \hat{c}'_3 \tag{151a, b}$$

It is convenient to employ dimensionless quantities instead of actual displacements, forces and couples:

$$\vec{v}_N \equiv \frac{\vec{V}_N}{L}, \qquad \Delta u_N \equiv \frac{\Delta U_N}{l}, \qquad \Delta v_N \equiv \frac{\Delta V_N}{l} \tag{152a, b, c}$$

$$x_N \equiv \frac{X_N}{L}, \qquad y_N \equiv \frac{Y_N}{L}, \qquad M \equiv \frac{L}{l} \tag{152d, e, f}$$

$$n_N \equiv \frac{N_N}{EA}, \qquad q_N \equiv \frac{l^2}{EI} Q_N, \qquad m_N \equiv \frac{l}{EI} M_N \tag{153a, b, c}$$

$$P \equiv \frac{l^2 P}{EI}, \qquad k \equiv \frac{Al^2}{I}, \tag{154), (155}$$

where E, A and I are the usual notations for the elastic modulus, cross-sectional area and moment-of-interia, respectively.

Since the linear analysis of the element is elementary, we cite the results: The constitutive equations are

$$\Delta u_N = \tfrac{1}{6} q_N - \tfrac{1}{2} m_N \tag{156a}$$

$$\Delta v_N = -n_N \tag{156b}$$

$$\Delta \psi_N = q_N - m_N \tag{156c}$$

Because we are concerned with finite rotations, we retain the product terms in the reactive conditions (131). Then the equilibrium conditions assume the forms:

$$q_{N+1} - q_N + k n_{N+1} \Delta \psi_N = 0 \tag{157a}$$

$$k(n_{N+1} - n_N) - q_{N+1} \Delta \psi_N = 0 \tag{157b}$$

$$m_{N+1} - m_N - q_N = 0 \tag{157c}$$

The conditions of continuity at the intersection of the Nth and $(N+1)$th element are

$$\vec{v}_N + M \overrightarrow{\Delta v}_N = \vec{v}_{N+1} \tag{158a}$$

$$\psi_N + \Delta \psi_N = \psi_{N+1} \tag{158b}$$

The boundary conditions at the left end of the beam are

$$m_1 = 0, \qquad k n_1 = p \cos \psi_1 \tag{159a}$$

$$q_1 = -p \sin \psi_1 \tag{159b}$$

The conditions at the right end of the beam are

$$\vec{v}_{M+1} = 0, \qquad \psi_{M+1} = 0 \tag{160a, b}$$

Equations (156) through (160) govern the behavior of the discrete system. The description of the individual element is exact in the context of linear beam theory and the description of the entire beam is valid under finite deflections.

TABLE 2

Number of Elements	2	3	4	8	11	22
p_e^*/p_E	1·621	1·268	1·149	1·035	1·019	1·005

The equations which determine the buckling load are obtained in the classical manner: Let $n_N^* = p^*/k$ denote the prebuckling value of n_N, δm_N the buckling increment of m_N, eliminate q_N from (157a, c) and (156c) and linearize in δm_N. The result is

$$\delta m_{N+2} - 2\delta m_{N+1} + \delta m_N - \tfrac{1}{2}P^*(\delta m_{N+1} + \delta m_N) = 0 \tag{161}$$

According to (157c) and (156c)

$$\delta(\Delta\psi_N) = \delta m_{N+1} - 2\delta m_N$$

Then, from (158b) we have

$$\delta\psi_{M+1} = \delta\psi_1 + \sum_{N=1}^{M} (\delta m_{N+1} - 2\delta m_N)$$

In view of the fixed-end condition (160b)

$$\delta\psi_1 = \sum_{N=1}^{M} (2\delta m_N - \delta m_{N+1}) \tag{162}$$

Using (157c) and (162), we obtain two boundary conditions from (159a, b) namely

$$\delta m_1 = 0 \tag{163a}$$

$$\delta m_2 + p^* \sum_{N=1}^{M} (2\delta m_N - \delta m_{N+1}) = 0 \tag{163b}$$

The difference equation (161) with $N = 1, \ldots, M-1$ and the end conditions (163a, b) provide a system of $(M+1)$ homogeneous equations in $(M+1)$ quantities δm_N. The determinant of the coefficients depends only on p^* and vanishes at the critical value p_e^*. Table 2 displays values of the ratio p_e^*/p_E where p_E is the critical value according to the Euler theory.

To treat the postbuckled deformations, we require the relations between the components of $\vec{v}_N$ and $\overrightarrow{\Delta v}_N$. In accordance with (158) and the definitions (150) and (152)

$$x_N = -M \sum_{L=N}^{M} (\Delta v_L \cos\psi_L - \Delta u_L \sin\psi_L) \tag{164a}$$

$$y_N = -M \sum_{L=N}^{M} [(1+\Delta v_L)\sin\psi_L + \Delta u_L \cos\psi_L] \tag{164b}$$

$$\psi_N = -\sum_{L=N}^{M} \Delta\psi_L \tag{164c}$$

The postbuckling deformation can be treated as follows: For any assigned load $p > p_e^*$, one assumes a value ψ_1 whereupon the quantities m_1, n_1, and q_1 are determined by the end conditions (159) and then Δu_1, Δv_1, and $\Delta\psi_1$ by the constitutive equations (156). Then

m_2, n_2 and q_2 are obtained by the simultaneous solution of (157). Next Δu_2, Δv_2 and $\Delta\psi_2$ are obtained from the constitutive equations (156), m_3, n_3, and q_3 from (157), etc., etc., until Δu_M, Δv_M and $\Delta\psi_M$ have been computed. The initial guess ψ_1 is correct only if $\psi_{M+1} = 0$. If the condition is not satisfied with sufficient precision, then the initial guess is revised and the computation repeated. When satisfactory precision is achieved the displacement and rotation at any station can be computed via (164) Numerical computations of ψ_1 obtained with 4 and 11 elements are shown in Fig. 8. The agreement with Euler's

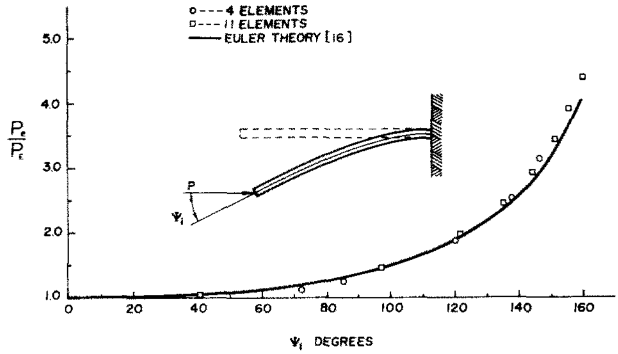

FIG. 8

theory of the "elastica" [16] is amazingly good. Table 3 presents typical values for the displacement components (x_1, y_1) at the end. As in our reference [16], shortening Δv_N is neglected in the latter computation.

TABLE 3. END DISPLACEMENT

$\psi_1 = 45{\cdot}9°$	$-x_1$	$-y_1$
8 Elements	0·837	0·486
Euler theory [16]	0·845	0·477

Illustrative example III

The merits of the simple approximations (110) and (111) for membranes are not likely to be disputed. Such approximations have proved adequate even for finite deformations [17]. The measure of these simple approximations together with the discrete Kirchhoff-type constraints is the ability to describe flexure. The bending of a plate provides another demonstration of this capability:

Numerical results were obtained for a square rectangular plate under uniform load (q) and central load (p), with simply supported (s) and clamped (c) edges. All computations were based on the same ratio ($h/a = 0{\cdot}100$) of thickness (h) to width (a). Non-dimensional values of the central deflection are given in Table 4. The number of elements is the number in a quadrant. Also shown in Table 4 are displacements computed by the Hermite interpolation [6] which provides 16 degrees-of-freedom for each element. The accuracy displayed by our simple approximation is surprisingly good since each element has only 8 degrees-of-freedom, yet maintains continuity at the interfaces of quadrilateral elements.

TABLE 4. RESULTS FOR SQUARE PLATES

Number of elements in each quadrant	Central deflection w/w_0							
	Bilinear polynomial with discrete constraints				Hermite interpolation 6 degree polynomial			
	$c-q$	$c-p$	$s-q$	$s-p$	$c-q$	$c-p$	$s-q$	$s-p$
1	—	—	0·7854	0·5496	1·002	0·9463	0·6221	0·8692
4	0·9613	0·8652	0·9775	0·9919	1·004	0·9794	0·8633	0·9472
9	0·9838	0·9378	0·9912	0·9925	1·000	0·9900	0·9246	0·9653
16	0·9926	0·9650	0·9954	0·9946	1·004	0·9964	—	—
25	0·9968	0·9777	0·9971	0·9959	—	—	—	—

s—simple supported, c—clamped, q—uniformly loaded, p—centrally loaded
w_0 denotes value of Kirchhoff theory

On applications of the theory

The motion of a thin elastic shell can be approximated by a discrete system of finite elements. The unknowns of the system are the generalized coordinates and the corresponding generalized forces. Six coordinates for each element can be identified as the global coordinates which define the gross motion; the remaining coordinates are the deformation coordinates which determine the strain field in accordance with a prior approximation. The Lagrange equations (122) for an element can be identified with gross motion (or equilibrium) or relative motion (constitutive equations). The compatibility requirements (128, 129) enforce continuity of the reference surface and contiguity of inter-element edges. The reactive conditions (131) fulfill Newton's law of action and reaction at inter-element boundaries. Kinematic or dynamic conditions at discrete points of the boundary complete the system. The analogy between the algebraic equations of the discrete system and the differential equations of a continuous shell is complete. The discrete system approaches the continuum as the size of the elements diminishes.

Our formulation of the discrete system is applicable to any shell provided that the physical components of strain are everywhere small compared to unity; the rotations and deflections may be arbitrarily large.

The equations of the discrete system are comparable to difference-approximations of the differential equations for the continuous shell. Indeed, as seen in our illustrative examples, suitable approximations lead to familiar difference-equations.

In some instances the finite-element approach appears to have an advantage because it admits geometrical insight based on the experience and intuition of the user. Moreover, by decomposing the motion into a rigid rotation and a small deformation, one can construct approximations for large deflections of shells based on the small relative deflections within small finite-elements; in some cases the approximation within the element may be exact in the context of linear elasticity. For example, if a thin rectangular sheet is turned into a cylinder, some portions undergo extreme rotations, although a narrow axial strip experiences small relative rotations. Other examples are the flexing of an annular segment to a cone and the axisymmetric deformation of a shallow cone [18].

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Абстракт—Перпдставляется концепция конечных элементов для анализа оболочек, с несколько важными успехами.

Вопервых, усовершенствовается теория Кирхгоффа, **путем** включния поперeченой деформации сдвига. усовершенствованная теория допускает более простые аппроксимирующие функции при сохранении непрерывности элементов сечения.

Во-вторых, разлагается движение элемента на движение жесткого тела, вызванное деформацией. Это разложение служит в целью использования существующих формул для упругих элементов для анализа задач, в которых учитываютсн конечные вращения и выпучивание.

Затем, приводятся уравнения Лагранжа для вывода уравнений дискретных систем. В результате примения этого метода получаются постоянные инерционные члены для любого рода движения, колебательного или нестационарного.

В заключение, приводятся самые простые аппроксимирующие полиномы, при учете теории деформации сдьга. Дальнейшие упрощения получаются путем введения сил сцепления анологично гнпотезе Кирхгоффа теории сплошной среды. Силы сцепления являются правильной базой для отсуствия влияния поперечного сдвига в энергии деформации. Результирующее приближение для указанных примеров явлется быстро сходимым к результатам теории Кирхгоффа.